Math 128a - Homework 5 - Due March 7

1) Complete the proof of the Weierstrass Approximation Theorem done in class, by proving the following two Lemmas:

1. Lemma 1: In class we showed that for any continuous function $f(x)$ on [0,1] with $f(0) = f(1) = 0$, and any $\epsilon > 0$ it is possible to find a polynomial $p(x)$ such that $|p(x) - f(x)| < \epsilon$ for all $0 \le x \le 1$. Use this to show that for any continuous function $g(x)$ on any finite interval $[a, b]$, with any finite values of $g(a)$ and $g(b)$, and any $\eta > 0$, it is possible to find a polynomial $q(x)$ such that $|q(x) - g(x)| < \eta$ for all $a \le x \le b$. Hint: consider $f(x) = g(x \cdot (b - a) + a) - g(a) - x(g(b) - g(a))$.

   Answer: Per the hint, given $g(x)$, let
$$f(x) = g(x(b-a)+a) - g(a) - x(g(b) - g(a))$$
   and check that $f(0) = 0 = f(1)$. Really we have used the following change of variables: $x = (y - a)/(b - a)$, $y = x(b - a) + a$, and we let $f(x) = g(y) - g(a) - x(g(b) - g(a))$, so that $g(y) = f(\frac{y-a}{b-a}) + g(a) + \frac{y-a}{b-a}(g(b) - g(a))$. The point is that this change of variables takes $\{a \le y \le b\}$ to $\{0 \le x \le 1\}$. Given $\epsilon > 0$, find $p(x)$ such that $|p(x) - f(x)| < \epsilon$ on $[0, 1]$ and let $q(y) = p(\frac{y-a}{b-a}) + g(a) + \frac{y-a}{b-a}(g(b) - g(a))$. Note that $q(y)$ is a polynomial in y, and:
$$\begin{aligned} |q(y) - g(y)| &= |[p(x) + g(a) + x(g(b) - g(a))] - [f(x) + g(a) + x(g(b) - g(a))]| \\ &= |p(x) - f(x)| \\ &< \epsilon \end{aligned}$$

2. Lemma 2: Let $c_n = \left[\int_{-1}^{1}(1 - x^2)^n dx\right]^{-1}$. Show that $c_n < \sqrt{n}$ when $n \ge 1$. Hint: Show that you can bound $\int_{-1}^{1}(1 - x^2)^n dx$ below by $\int_{-1/\sqrt{n}}^{1/\sqrt{n}}(1 - x^2)^n dx$. Then show that you can bound the integrand below by $1 - nx^2$ on the interval of integration.

   Answer: Let $d_n = \int_{-1}^{1}(1 - x^2)^n dx$ so that $c_n = 1/d_n$. We will show that $d_n > 1/\sqrt{n}$ and thus that $c_n < \sqrt{n}$. First of all, since $n \ge 1$ (otherwise the lemma is false), $[-1/\sqrt{n}, 1/\sqrt{n}] \subset [-1, 1]$, so $d_n > e_n = \int_{-1/\sqrt{n}}^{1/\sqrt{n}}(1 - x^2)^n dx$ (this also depends on the fact that $(1 - x^2)^n > 0$ which is true because $x \in [0, 1]$ by assumption). Now on this domain of integration, we have $x^2 < 1/n$, so by the Binomial Theorem
$$(1 - x^2)^n = 1 - nx^2 + \sum_{i=2}^{n} \frac{n!}{(i!)(n-i)!}(-1)^i x^{2i}$$
   Thus, to show that $(1 - x^2)^n > 1 - nx^2$ on the domain of integration we need to show that
$$\sum_{i=2}^{n} \frac{n!}{(i!)(n-i)!}(-1)^i (x^2)^i \ge 0 \tag{1}$$
   on the domain. The important point is that, on $[-1/\sqrt{n}, 1/\sqrt{n}]$, $0 \le x^2 \le 1/n$. The sum (1) which we want to assure is positive starts with a positive term and either has the same number of negative terms as positive terms or one more positive term. Thus we want to show that, for each $i \ge 2$:
$$\frac{n!}{(i!)(n-i)!}(x^2)^i \ge \frac{n!}{(i+1)!(n-i-1)!}(x^2)^{i+1}$$
   because if we do this then each positive term will dominate the succeeding negative term and the total sum will be positive. The above inequality is equivalent to:
$$\frac{i+1}{n-i} \ge x^2$$

which is true because $x^2 \le 1/n$ and $1 < i+1$ while $n > n-i$.

Thus we can conclude that

$$\begin{aligned} d_n &> \int_{-1/\sqrt{n}}^{1/\sqrt{n}} (1-nx^2)dx \\ &= \frac{4}{3}\frac{1}{\sqrt{n}} \\ &> \frac{1}{\sqrt{n}} \end{aligned}$$

2) We will use the proof of the Weierstrass Approximation Theorem to show how to bound the degree of the polynomial needed to approximate $f(x) = \sin(\pi x)$ on [0,1] to within any $\eta > 0$. In other words, your answer will be a function $g(\eta)$ (implemented as a program) with the following property: If $n \ge g(\eta)$, then it is possible to find a polynomial $p(x)$ of degree at most n such that $|p(x) - \sin(\pi x)| \le \eta$ for $0 \le x \le 1$. Are your computed values of n much larger than needed to find a polynomial of error η, or about right? Is the polynomial constructed in the proof of the Weierstrass Approximation Theorem a good one to use in practice? Hint: First, find an explicit value for $M \ge |\sin(\pi x)|$ in Lemma 3 in the proof of the Weierstrass Approximation Theorem in the class notes. Second, find an explicit value for δ in Lemma 3, which will depend on $\epsilon = \eta/2$ and have the property that $|x-y| < \delta$ implies $|\sin(\pi x) - \sin(\pi y)| < \eta/2$. How steep can $\sin(\pi x)$ be? Finally, write a simple program that computes an n that guarantees that $(1-\delta)2M\sqrt{n}(1-\delta^2)^n < \eta/4$. The output of your program is $g(\eta)$. Tabulate $g(\eta)$ for $\eta = 10^k$, $k = -5, -10, -15, -20, -25$. Your program should return as small a value of $g(\eta)$ as you can guarantee is correct.

Answer: First, $M = 1 \le |\sin(\pi x)|$ everywhere. Second, $|f'(x)| = \pi|\cos(\pi x)| \le \pi$ everywhere, and by the mean value theorem $|f(x) - f(y)| = |f'(c)||x-y|$ for some point c between x and y, so $|f(x) - f(y)| \le \pi|x-y|$ for any x, y. Thus if $\delta = \eta/(2\pi)$ then whenever $|x-y| < \delta$, we will have $|f(x) - f(y)| < \eta/2$. You might try to run the following matlab program to find n:

```
function [bestn] = g(eta)
M=1;delta=eta/(2*pi);n=1;
while((1-delta)*2*M*sqrt(n)*(1 - delta^2)^n >= eta/4),
    n=n+1;
end;
bestn = n;
```

BUT, since $fl(1-\delta^2) = 1$ when δ is very small, this won't work. We will instead use the following approximation (the first limit can be proved with L'Hopital's rule):

$$\begin{aligned} \lim_{x\to 0}(1-x)^{\frac{1}{x}} &= \frac{1}{e} \quad \text{so, for } x = \delta^2 \text{very small :} \\ (1-\delta^2)^n &= ((1-x)^{\frac{1}{x}})^{nx} \\ &\approx (\tfrac{1}{e})^{n\delta^2}) \quad \text{so :} \\ (1-\delta)2M\sqrt{n}(1-\delta^2)^n &\approx 2n^{1/2}e^{-n\delta^2} \quad \text{so we want :} \\ n^{1/2}e^{-n\delta^2} &< \frac{\eta}{2} \quad \text{or :} \\ -n\delta^2\frac{1}{2}\ln(n) &< \ln\frac{\eta}{2} \quad \text{or :} \\ n\ln n &> -2\frac{\ln\frac{\eta}{2}}{\delta^2} \end{aligned}$$

Now try this program:

```
% function [bestn] = best(eta,start,inc)
% find as small an n as possible with
% n*log(n)>-2*log(eta/2)/(delta^2)
% where delta = eta/(2*pi)
%
% inputs: eta
% start = n value to start at
% inc = initial increment; program will succesively halve
% this to get better estimates for n
% if inc = 1 this goes very slowly, so best to
% start with a large increment
% outputs: bestn = best n found
%
function [bestn] = best(eta,start,inc)

delta=eta/(2*pi);
n=start;
stop = -2*log(eta/2)/(delta^2)
while (inc > 0)
      if ((n+inc)*log(n+inc) > stop)
            inc = floor(inc/2);
      else
            nnew = n+inc;
%           Guard against n+inc rounding to n and getting into an infinite loop
            if (n == nnew)
                  inc = 0;
            end
            n=nnew;
      end
end
bestn = n + 1;
```

Results:

eta	n
1e-5	3.621061459540000e+11
1e-10	3.769585013558981e+21
1e-15	3.825266269443624e+31
1e-20	3.854437985739969e+41
1e-25	3.872392907403787e+51

Note how enormous n has to be, according to this result! In practice, we know that n can be much smaller (and will study this more in lecture).

3) Problem 6.1.33

Answer: Let $g_0(x)$, $g_1(x)$, ... , $g_n(x)$ be a basis of the linear space E. This means any function in E can be written as a linear combination $\sum_{i=0}^{n} c_i g_i(x)$ for some unique values of c_i.

Now assume there exists a unique solution $c_0, ..., c_n$ of $\sum_{i=1}^{n} c_i g_i(x_j) = y_j$, $j = 0, ..., n$, for any values of y_j. Then if all $y_j = 0$, $c_i = 0$ is the unique solution, and so $0 = \sum_{i=1}^{n} c_i g_i(x)$ is the unique function in E that is 0 at all x_j.

Conversely, suppose that $0 \in E$ is the only function that equals 0 at all x_j. This means that the determinant of the matrix

$$G = \begin{bmatrix} g_0(x_0) & g_1(x_0) & \cdots & g_n(x_0) \\ g_0(x_1) & g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ g_0(x_n) & g_1(x_n) & \cdots & g_n(x_n) \end{bmatrix}$$

is nonzero, because otherwise there would be a nonzero null vector

$$c = \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix}$$

such that $Gc = 0$, which means there would be a nonzero function $\sum_{i=0}^{n} c_i g_i(x) \in E$ that is zero at all the x_j, a contradiction. Since G is nonsingular, we can solve

$$Gc = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

to get a unique solution c for any y_j, as desired.

4) Problem 6.1.34.

Answer: *Answer:* Let V be $n+1$-by-$n+1$ matrix with entries $V_{ij} = x_i^j$, $0 \le i, j \le n$. V's determinant is a polynomial in its entries. We will determine this polynomial by determining all its factors. By considering the expansion of $\det V$ along the first row, we see that considered as a polynomial in x_0, its degree is n. Similarly, by expanding along other rows, its degree in any x_i is n. Now note that if any $x_i = x_j$, rows i and j are identical, so $\det(V) = 0$. This means that $x_j - x_i$ is a factor of $\det(V)$, since $x_j - x_i = 0$ implies $\det(V) = 0$. Thus all the $n(n+1)/2$ distinct factors $x_j - x_i$, $0 \le i < j \le n$, are factors, and so $\mathrm{P} = \prod_{0 \le i < j \le n}(x_j - x_i)$ is a factor of $\det(V)$. Considering P as a polynomial in x_i for some fixed i, we see its degree is n, because x_i appears in $n - i$ factors $x_j - x_i$ (for $j > i$) and in i factors $x_i - x_k$ (for $k < i$). Thus P divides $\det(V)$ and has the same degree in all variables, so $\det V = \alpha P$ for some constant α that we have to determine. We show that $\alpha = 1$ by induction on n. Assume, by induction, that $\det(V_{n-1}) = P_{n-1}$, where the subscript denotes the dimension of V. Expanding $\det(V_n)$ by the last column, we see that the coefficient of x_n^n in $\det(V_n)$ is precisely $\det(V_{n-1}) = P_{n-1}$. But one can see that the coefficient of x_n^n in P_n is also P_{n-1}, so $\alpha_n = 1$.