Math 128a - Homework 5 - Due March 7

1) Complete the proof of the Weierstrass Approximation Theorem done in class, by proving the following two Lemmas:

1. Lemma 1: In class we showed that for any continuous function \( f(x) \) on \([0,1]\) with \( f(0) = f(1) = 0 \), and any \( \epsilon > 0 \) it is possible to find a polynomial \( p(x) \) such that \( |p(x) - f(x)| < \epsilon \) for all \( 0 \leq x \leq 1 \). Use this to show that for any continuous function \( g(x) \) on any finite interval \([a, b]\), with any finite values of \( g(a) \) and \( g(b) \), and any \( \eta > 0 \), it is possible to find a polynomial \( q(x) \) such that \( |q(x) - g(x)| < \eta \) for all \( a \leq x \leq b \). Hint: consider \( f(x) = g(x \cdot (b-a)+a) - g(a) - x(g(b) - g(a)) \).

2. Lemma 2: Let \( c_n = \left[ \int_{-1}^{1} (1-x^2)^n \, dx \right]^{-1} \). Show that \( c_n < \sqrt{n} \) when \( n \geq 1 \). Hint: Show that you can bound \( \int_{-1}^{1} (1-x^2)^n \, dx \) below by \( \int_{-1}^{1/\sqrt{n}} (1-x^2)^n \, dx \). Then show that you can bound the integrand below by \( 1 - nx^2 \) on the interval of integration.

2) We will use the proof of the Weierstrass Approximation Theorem to show how to bound the degree of the polynomial needed to approximate \( f(x) = \sin(\pi x) \) on \([0,1]\) to within any \( \eta > 0 \). In other words, your answer will be a function \( g(\eta) \) (implemented as a program) with the following property: If \( n \geq g(\eta) \), then it is possible to find a polynomial \( p(x) \) of degree at most \( n \) such that \( |p(x) - \sin(\pi x)| \leq \eta \) for \( 0 \leq x \leq 1 \). Are your computed values of \( n \) much larger than needed to find a polynomial of error \( \eta \), or about right? Is the polynomial constructed in the proof of the Weierstrass Approximation Theorem a good one to use in practice? Hint: First, find an explicit value for \( M \geq |\sin(\pi x)| \) in Lemma 3 in the proof of the Weierstrass Approximation Theorem in the class notes. Second, find an explicit value for \( \delta \) in Lemma 3, which will depend on \( \epsilon = \eta/2 \) and have the property that \( |x - y| < \delta \) implies \( |\sin(\pi x) - \sin(\pi y)| < \eta/2 \). How steep can \( \sin(\pi x) \) be? Finally, write a simple program that computes an \( n \) that guarantees that \((1-\delta)2M\sqrt{n}(1-\delta^2)^n < \eta/4 \). The output of your program is \( g(\eta) \). Tabulate \( g(\eta) \) for \( \eta = 10^k, k = -5, -10, -15, -20, -25 \). Your program should return as small a value of \( g(\eta) \) as you can guarantee is correct.

3) Problem 6.1.33

4) Problem 6.1.34.