**Math 128a - Homework 5 - Due March 7**

1) Complete the proof of the Weierstrass Approximation Theorem done in class, by proving the following two Lemmas:

1. Lemma 1: In class we showed that for any continuous function $f(x)$ on [0,1] with $f(0) = f(1) = 0$, and any $\epsilon > 0$ it is possible to find a polynomial $p(x)$ such that $|p(x) - f(x)| < \epsilon$ for all $0 \leq x \leq 1$. Use this to show that for any continuous function $g(x)$ on any finite interval $[a, b]$, with any finite values of $g(a)$ and $g(b)$, and any $\eta > 0$, it is possible to find a polynomial $q(x)$ such that $|q(x) - g(x)| < \eta$ for all $a \leq x \leq b$. Hint: consider $f(x) = g(x \cdot (b-a) + a) - g(a) - x(g(b) - g(a))$.
2. Lemma 2: Let $c_n = \left[\int_{-1}^{1}(1-x^2)^n dx\right]^{-1}$. Show that $c_n < \sqrt{n}$ when $n \geq 1$. Hint: Show that you can bound $\int_{-1}^{1}(1-x^2)^n dx$ below by $\int_{-1/\sqrt{n}}^{1/\sqrt{n}}(1-x^2)^n dx$. Then show that you can bound the integrand below by $1 - nx^2$ on the interval of integration.

2) We will use the proof of the Weierstrass Approximation Theorem to show how to bound the degree of the polynomial needed to approximate $f(x) = \sin(\pi x)$ on [0,1] to within any $\eta > 0$. In other words, your answer will be a function $g(\eta)$ (implemented as a program) with the following property: If $n \geq g(\eta)$, then it is possible to find a polynomial $p(x)$ of degree at most $n$ such that $|p(x) - \sin(\pi x)| \leq \eta$ for $0 \leq x \leq 1$. Are your computed values of $n$ much larger than needed to find a polynomial of error $\eta$, or about right? Is the polynomial constructed in the proof of the Weierstrass Approximation Theorem a good one to use in practice? Hint: First, find an explicit value for $M \geq |\sin(\pi x)|$ in Lemma 3 in the proof of the Weierstrass Approximation Theorem in the class notes. Second, find an explicit value for $\delta$ in Lemma 3, which will depend on $\epsilon = \eta/2$ and have the property that $|x - y| < \delta$ implies $|\sin(\pi x) - \sin(\pi y)| < \eta/2$. How steep can $\sin(\pi x)$ be? Finally, write a simple program that computes an $n$ that guarantees that $(1-\delta)2M\sqrt{n}(1-\delta^2)^n < \eta/4$. The output of your program is $g(\eta)$. Tabulate $g(\eta)$ for $\eta = 10^k$, $k = -5, -10, -15, -20, -25$. Your program should return as small a value of $g(\eta)$ as you can guarantee is correct.

3) Problem 6.1.33

4) Problem 6.1.34.