Math 128a - Homework 4 - Due Feb 28 at the beginning of class

1) Section 3.3, Problem 8

2) You have just had a new baby, and realize that you have to start saving a lot of money to afford to send the child to college. Planning for the worst case, you ask yourself how much you would have to save to send your child to Stanford in 18 years. Assuming 7% inflation for college tuition (a widely recommended number for concerned parents), you compute that the current \$30,000/year cost of Stanford (out-of-date cost, but good enough for this question) will rise to $\$30000 \cdot 1.07^{18} \approx \$101398$ per year, or \$405592 for four years. By eating a lot of macaroni and cheese, you figure you can save \$10000/year to invest for the next 18 years. Assume that all your investment income will be taxed at a 20% rate at the end of 18 years. What interest rate do you need to get to have saved \$405592, after taxes, after 18 years? Given that the historic rate of return of the stock market is around 10%, do you expect to have saved enough? Hint: Use the formulation from the last question. Note that you pay tax only on the interest you earn, not the \$180000 you invest (which would be double taxation). Explain how you got your answer; "I plugged it into an HP-12C financial calculator" is not good enough. We note that the solver for this problem on the HP-12C was designed by Prof. W. Kahan in our department.

3) Consider the following system of two simultaneous equations in two unknowns x and y:

$$\begin{aligned} f_1(x,y) = x^2 + a \cdot y^2 - 1 &= 0 \\ f_2(x,y) = (x-1)^2 + y^2 - 1 &= 0 \end{aligned}$$

where a is a parameter.

Part 1. Solve the equations explicitly by hand. Show that for all but a finite number of (complex) values of a, there are four (possibly complex) solutions (x, y), but for this finite set of a's, there are fewer solutions. Exhibit all solutions, and the finite set of a's, explicitly. Four is the upper bound on the number of solutions given by Bezout's theorem as described in class (the product of the degrees of the polynomials f_1 and f_2, which are both 2). This example shows that the upper bound is attainable for most, but not necessarily all, coefficients.

Part 2. Write down two explicit formulas for Newton iteration for this system. First, write it in the form

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} - J^{-1} \cdot \begin{bmatrix} x_i^2 + a \cdot y_i^2 - 1 \\ (x_i - 1)^2 + y_i^2 - 1 \end{bmatrix}$$

where you explicitly exhibit the Jacobian J and its inverse J^{-1}. Second, evaluate this expression explicitly, i.e. multiply it out and simplify.

Part 3. Using these formulas, write a (very short) Matlab program to implement Newton iteration just for this example. It need only implement one step, i.e. not test test convergence, so that you can just run it "by hand" and look at the iterates to determine convergence. Thus it need only be a few lines long at most.

1. Try it for each of the finite values of a that only have two solutions, and starting guesses equal to the true solutions times (1+rand(1)/100), i.e. change the true solution by about 1%. How many steps does it take to converge to machine precision? Do the number of correct digits roughly double at each step?

2. Try it for 3 random complex choices of a (=randn(1)+sqrt(-1)*randn(1)), and again take the true solutions, perturb them randomly by 1%, and see if Newton converges quadratically.