Math 128a - Homework 3 - Due Sept 21 at the beginning of class

1) Sometimes is difficult or expensive to compute $f'(x_n)$ for use in Newton's method, so we use an approximation α instead. Find the condition on α to ensure that the iteration $x_{n+1} = x_n - f(x_n)/\alpha$ will converge (at least) linearly to a zero of f if started near enough to the zero.

2) Show that when $f(z) = f'(z) = 0 \neq f''(z)$, i.e. z is a double zero, then Newton's method only converges linearly to z (with constant .5) when started close enough to z, not quadratically. Show that if z is a double zero, then the modified Newton iteration $x_{n+1} = x_n - 2 \cdot f(x_n)/f'(x_n)$ is quadratically convergent. (Note: A similar argument shows that if x is a root of multiplicity k, then $x_{n+1} = x_n - k \cdot f(x_n)/f'(x_n)$ converges quadratically, but you do not have to show this.)

3) Suppose we want to implement the squareroot function using Newton's method $x_{n+1} = (x_n + a/x_n)/2$ to solve $f(x) = x^2 - a = 0$. The input a can be any positive normalized IEEE double precision number (zero and subnormal numbers are handled as special cases). We want to quickly find a starting value x_1 that guarantees that 3 Newton steps are enough to get nearly full precision (relative error near 2^{-52}) no matter what a is. We will compute x_1 with a little bit-fiddling and a table as follows:

Part 1. Using the floating point representation $a = 2^e(1 + f)$, where $0 \le f < 1$, rewrite a as $a = 2^{2e'}(1 + f')$, where $e' = \lfloor e/2 \rfloor$ (i.e. $e/2$ rounded down to the nearest integer) and f' is modified appropriately. What range can $1 + f'$ lie in?

Part 2. Since $a^{1/2} = 2^{e'}(1 + f')^{1/2}$, it suffices to apply Newton to $1 + f'$. We will pick a starting value x_1 by looking it up in a table. Write $1 + f' = bb.bbbbb..._2$, i.e. as a binary number with a fractional part, where each b is a bit. Suppose we take the leading k bits of this number (which approximate $1 + f'$, the better for larger k), and look up its square root $x(1)$ in a table (k bits means we need a table of 2^k values, which we index into just by interpreting the leading k bits as a binary integer). What is the smallest possible k that guarantees $x(1)$ is accurate enough to guarantee that we only need 3 Newton iterations? 2 Newton iterations? Do not worry about roundoff in your analysis. (In actual IEEE arithmetic, which requires the square root to be correctly rounded, extra care in the algorithm is needed to ensure this, but this algorithm is nearly accurate enough, and has been frequently used in the past.)

4) Prove that, if $F : [a, b] \to \mathbf{R}$, if F' is continuous and if $|F'(x)| < 1$ on $[a, b]$ then F is a contraction on $[a, b]$. Does F necessarily have a fixed point?

5) Prove that if F is a continuous map of $[a, b]$ into $[a, b]$ then F must have a fixed point. Then determine if this assertion is true for functions from $\mathbf{R}$ to $\mathbf{R}$.