Math 128a - Midterm 1

This exam is open book, open notes, open calculator (you shouldn't need one). The total score is 50 pts. The number of points approximately indicates the number of minutes you should spend on the problem.

1) (8 pts) Let $f(x) = e^x + x - 7$. Using a result proven in class, show that Newton's method converges to a zero of $f(x)$ starting from $x_0 = -18$. Don't forget to show that $f(x)$ actually has a zero.

Answer: $f(0) = -6 < 0$ and $f(7) = e^7 > 0$, so $f(x)$ has a zero between 0 and 7 by the Intermediate Value Theorem. $f'(x) = e^x + 1 > 0$ for all x, so $f(x)$ is strictly increasing, and thus the zero is unique. $f''(x) = e^x > 0$ for all x, so the function is convex. Therefore, by Theorem 2 on page 91 of the text, Newton converges from *any* starting point, inluding $x_0 = -18$.

2) (8 pts) Consider the polynomial $p(x) = x^3 - 3x^2 + 3x - 1$. If I make a relative change in just the constant term of at most 10^{-15}, how large a relative change in the roots of $p(x)$ can that cause? Hint: Just argue from "first principles," rather than using a general result from class.

Answer: $p(x) = (x-1)^3$ has three zeros at 1. Perturbing the constant term by a relative change of at most 10^{-15} changes $p(x)$ to $p(x) - \epsilon$, where $|\epsilon| \leq 10^{-15}$. This changes the roots to $1 + \epsilon^{1/3}$, where $\epsilon^{1/3}$ is a complex number of magnitude at most $(10^{-15})^{1/3} = 10^{-5}$. This changes the roots relatively by at most 10^{-5}.

3) (14 pts) Let x_0 be any IEEE double precision floating point number between 1 and 2, inclusive. Define $x_i = fl(1 + 4 * (x_{i-1} - 1.5)^2)$, where $fl()$ means the floating point result of executing the expression in parentheses, rounding after every arithmetic operation using the rules of IEEE arithmetic.

1. (7 pts) Show that the floating point number x_i always lies in the interval $[1, 2]$. Hint: Start by showing this is true in exact arithmetic.

   Answer: If x_{i-1} is between 1 and 2, $x_{i-1} - 1.5$ is between $-.5$ and $.5$ in exact arithmetic. Since 1.5, $1.5 - 1 = .5$ and $1.5 - 2 = -.5$ are exact floating point numbers, the monotonity of floating point arithmetic means that $fl(x_{i-1} - 1.5)$ is between $-.5$ and $.5$. Since $.5^2 = .25$ and $0^2 = 0$ are also exact floating point numbers, monotonicity again means that $fl((x_{i-1} - 1.5)^2)$ is between 0 and .25. Since 4, $4 * .25 = 1$, and $4 * 0 = 0$ are exact floating point numbers, monotonicity again means $fl(4 * (x_{i-1} - 1.5)^2)$ is between 0 and 1. Finally, since 1, $1 + 0 = 0$ and $1 + 1 = 2$ are exact floating point numbers, monotonicity again implies that $x_i = fl(1 + 4 * (x_{i-1} - 1.5)^2)$ is between 1 and 2.

2. (7 pts) Show that the sequence is eventually periodic, i.e. there are some integers $i < j$ such that $x_i = x_j$, and that the sequence $x_i, x_{i+1}, x_{i+2}, ..., x_{j-1}$ repeats endlessly.

   Answer: Since all the x_i are between 1 and 2, and there are a finite number of floating point numbers in that range, the Pigeon Hole Principle means that they eventually repeat. Since the formula for x_i in terms of x_{i-1} is deterministic, they sequence will repeat itself exactly every time it returns to x_i.

4) (20 pts) Both the Weierstrass Approximation Theorem and Taylor's Theorem give conditions on when a function $f(x)$ can be approximated to within any $\epsilon > 0$ by a polynomial (in the latter case, by expanding $f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(c)}{j!}(x-c)^j$ around some c, taking the real part if c is complex, and truncating this infinite expansion). In this question, we will explore the differences between these Theorems.

For each of the following functions $f(x)$ defined on an interval I, answer the following two questions:

1. Can you use Weierstrass's Theorem to prove that we can approximate $f(x)$ by a polynomial to within any $\epsilon > 0$ on I? Explain very briefly why or why not.

2. Can you use Taylor's Theorem to prove that we can approximate $f(x)$ by a polynomial to within any $\epsilon > 0$ on I? Explain very briefly why or why not.

Here is the list of functions and intervals:

1. (6 pts) $f(x) = |x|$ on $I = [-1, 2]$. Hint: If $f(x)$ has a Taylor expansion that converges in any neighborhood around some $x = z$, then $f(x)$ is differentiable at z.

   Answer: $f(x)$ is continuous on the closed bounded interval I, so Weierstrass implies that we can approximate it to within any $\epsilon > 0$ by a polynomial. Since $f(x)$ is not differentiable at $0 \in I$, we cannot use Taylor's Theorem to expand it in a region including any neighborhood of 0, by the hint.

2. (6 pts) $f(x) = 1/x$ on $I = [1, 2]$. Hint: Can you write $1/x$ as a geometric series?

   Answer: $f(x)$ is continuous on the closed bounded interval I, so Weierstrass implies that we can approximate it to within any $\epsilon > 0$ by a polynomial. We may write $1/x = 1/(2-(2-x)) = .5/(1-(2-x)/2) = \sum_{i=0}^{\infty}(2-x)^i/2^{i+1}$, which is a geometric series converging for $|2-x|/2 < 1$, or $0 < x < 4$, which certainly includes I. This is a Taylor expansion of $1/x$ around 2, since Taylor expansions are unique. By truncating this Taylor expansion at term $(2-x)^n/2^{n+1}$, we can approximate $f(x)$ as accurately as we want, because the remainder $\sum_{i=n+1}^{\infty}(2-x)^i/2^{i+1} = (2-x)^{n+1}/(2^{n+2}x)$ is bounded by $1/2^{n+2}$ for $x \in I$, which goes to zero.

3. (8 pts) $f(x) = 1/(1+x^2)$ on $I = [-2, 2]$. Hint: If a Taylor expansion around c (i.e. $f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(c)}{j!}(x-c)^j$) converges for $x = z$, then it converges for all complex x closer to c than z, i.e. for all complex x in the open disk with center c and radius $|z-c|$.

   Answer: $f(x)$ is continuous on the closed bounded interval I, so Weierstrass implies that we can approximate it to within any $\epsilon > 0$ by a polynomial. Now suppose we had a convergent Taylor series for $f(x)$ centered around some c, that also converged for all $x \in I$. Then by the hint, it would converge at all points in a disk containing I. But a disk containing I must also contain at least one of the points $\sqrt{-1} = i$ or $-i$. (For suppose c is in the first quadrant of the complex plane; then c is closer to i than -2 so the disk contains i. The other quadrants are similar.) But then the Taylor series would converge at $+i$ or $-i$, which is impossible, since $f(x)$ has a singularity there.