Math 128a - Midterm 1

This exam is open book, open notes, open calculator (you shouldn't need one). The total score is 50 pts. The number of points approximately indicates the number of minutes you should spend on the problem.

1) (8 pts) Let $f(x) = e^x + x - 7$. Using a result proven in class, show that Newton's method converges to a zero of $f(x)$ starting from $x_0 = -18$. Don't forget to show that $f(x)$ actually has a zero.

2) (8 pts) Consider the polynomial $p(x) = x^3 - 3x^2 + 3x - 1$. If I make a relative change in just the constant term of at most 10^{-15}, how large a relative change in the roots of $p(x)$ can that cause?

3) (14 pts) Let x_0 be any IEEE double precision floating point number between 1 and 2, inclusive. Define $x_i = fl(1 + 4 * (x_{i-1} - 1.5)^2)$, where $fl()$ means the floating point result of executing the expression in parentheses, rounding after every arithmetic operation using the rules of IEEE arithmetic.

1. (7 pts) Show that the floating point number x_i always lies in the interval $[1, 2]$. Hint: Start by showing this is true in exact arithmetic.
2. (7 pts) Show that the sequence is eventually periodic, i.e. there are some integers $i < j$ such that $x_i = x_j$, and that the sequence $x_i, x_{i+1}, x_{i+2}, ..., x_{j-1}$ repeats endlessly.

4) (20 pts) Both the Weierstrass Approximation Theorem and Taylor's Theorem give conditions on when a function $f(x)$ can be approximated to within any $\epsilon > 0$ by a polynomial (in the latter case, by truncating the Taylor Expansion). In this question, we will explore the differences between these Theorems.

For each of the following functions $f(x)$ defined on an interval I, answer the following two questions:

1. Can you use Weierstrass's Theorem to prove that we can approximate $f(x)$ by a polynomial to within any $\epsilon > 0$ on I? Explain very briefly why or why not.
2. Can you use Taylor's Theorem to prove that we can approximate $f(x)$ by a polynomial to within any $\epsilon > 0$ on I? Explain very briefly why or why not.

Here is the list of functions and intervals:

1. (4 pts) $f(x) = |x|$ on $I = [-1, 2]$
2. (4 pts) $f(x) = 3 \cdot x^{10} - \pi \cdot x^2 + 7$ on $I = (-\infty, \infty)$
3. (4 pts) $f(x) = 1/x$ on $I = [1, 2]$
4. (4 pts) $f(x) = 1/(1 + x^2)$ on $I = [-2, 2]$
5. (4 pts) $f(x) = e^{-1/x^2}$ on $I = [-1, 1]$