Math 128a - Final Exam - Fall 1998

This exam is open book, open notes, open homework, open calculator (you shouldn't need one). You can use any result from the class notes, text of the book (not unassigned problems!) or homework. The total score is 150 pts. The number of points approximately indicates the number of minutes you should spend on the problem (with 30 minutes left at the end for checking your work).

1) (35 points) We will consider how to use polynomial interpolation to implement the function $\exp(x) = e^x$ available on all computers. Most algorithms use the identity $e^x = 2^z$ where $z = (\log_2 e) \cdot x$, so we will just concentrate on the following algorithm for $y = 2^z$:

1)	$z1 = \lfloor z \rfloor$	... i.e. rounded down to the nearest integer
2)	$z2 = z - z1$	... so $0 \le z2 < 1$ and $2^z = 2^{z_1 + z_2} = 2^{z_1} \cdot 2^{z_2}$
3)	$s1 = 2^{z1}$	
4)	$s2 = 2^{z2}$	
5)	$y = s1 * s2$	

One can show that the only source of error is step 4), which we will concentrate on.

1. (25 points) Step 4) is typically implemented using polynomial approximation. We will use polynomial interpolation at the points $0 \le z_0 < z_1 < \cdots < z_n \le 1$. Let $p_n(z)$ denote the polynomial interpolant for 2^z at these points. Give explicit expressions for the z_i that guarantee the error bound

$$\max_{0 \le z \le 1} |p_n(z) - 2^z| \le \frac{1}{(n+1)!} \cdot (\ln 2)^{n+1} \cdot 2^{2-2n}$$

2. (10 points) Using the inequalities $\ln 2 < 2^{-1/2}$ and $(n+1)! > 2^{2n}$ (true for $n \ge 6$), how big does n have to be to guarantee an approximation good to double precision, that is $|2^z - p_n(z)| \le 2^{-53}$. Consider only the error from interpolation, not roundoff or other possible sources.

2) (25 points) Here we consider cubic splines.

1. (15 points) Determine what constraints a_i, b_i, and c_i, for $i = 1, 2, 3$, must satisfy for $f(x)$ to be a cubic spline. Your answer should consist of equations defining or relating these parameters.

$$f(x) = \begin{cases} a_1(x-4)^2 + a_2(x-3)^3 + a_3(x-2)^4 & x \leq 3 \\ b_1(x-4)^2 + b_2(x-4)^3 + b_3(x-4)^4 & 3 \leq x \leq 5 \\ c_1(x-4)^2 + c_2(x-5)^3 + c_3(x-5)^4 & 5 \leq x \end{cases}$$

2. (10 points) Determine the values of the parameters a_i, b_i, and c_i so that the cubic spline interpolates the points $(x_0, y_0) = (3, 5)$, $(x_1, y_1) = (0, -1)$ and $(x_2, y_2) = (6, 21)$.

3) (20 points) Determine the truncation errors for the following 2 approximations of $f^{(3)}(x)$, and determine which is more accurate. The truncation error should be in the form (something depending on f)$\cdot h^c + O(h^{c+1})$.

$$\begin{aligned} p_1(x) &= \frac{1}{h^3}[f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)] \\ p_2(x) &= \frac{1}{2h^3}[f(x+2h) - 2f(x+h) + 2f(x-h) + f(x-2h)] \end{aligned}$$

4) (35 points) Consider the implicit second order integration formula for $x'(t) = f(x(t))$: $x_{n+1} = x_n + \frac{h}{2}(f(x_{n+1}) + f(x_n))$, $h > 0$. Consider applying this formula to the differential equation $x'(t) = \lambda x(t)$ where λ is a constant. λ may be any complex number $\lambda = \lambda_r + i \cdot \lambda_i$, where $i = \sqrt{-1}$ and λ_r and λ_i are real.

1. (5 points) Write down an explicit expression for x_n (the numerical solution from the formula) in terms of $x_0 = x(0)$.
2. (5 points) Write down an explicit expression for $x(t)$ (the true solution) in terms of $x(0)$.
3. (5 points) Under what conditions on λ does $\lim_{n\to\infty} |x_n| = 0$ for any $x_0 \neq 0$?
4. (5 points) Under what conditions on λ does $\lim_{n\to\infty} |x_n| = \infty$ for any $x(0) \neq 0$?
5. (5 points) Under what conditions on λ does $\lim_{t\to\infty} |x(t)| = 0$ for any $x_0 \neq 0$?
6. (5 points) Under what conditions on λ does $\lim_{t\to\infty} |x(t)| = \infty$ for any $x(0) \neq 0$?
7. (5 points) Show that $\lim_{n\to\infty} |x_n| = \lim_{t\to\infty} |x(t)|$ for any $x(0) = x_0$.

5) (35 points) This question is about computing null vectors, i.e. nonzero vectors x satisfying $A \cdot x = 0$.

1. (15 points) Suppose $A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ is an n-by-n matrix where A_1 is i-by-i and nonsingular, A_2 is i-by-$n-i$ and the last $n-i$ rows are zero. Then it is a fact that the rank of A is i, so that there are $n-i$ linearly independent null vectors $x_1,...,x_{n-i}$. Given an algorithm for computing $x_1,...,x_{n-i}$. Note that they are not uniquely defined; any linearly independent set of $n-i$ null vectors will do. Your algorithm should be expressed at the level of "Factor the matrix B into $B = PLU$ using Gaussian elimination with partial pivoting" or "Solve the lower triangular system $Lx = y$ using forward substitution" rather than writing out loops in detail. Be sure to show that your vectors are independent.

2. (10 points) Let A be an n-by-n matrix. Show that if A has rank i, then at step $i+1$ of Gaussian elimination with complete pivoting, the largest entry found in the submatrix $A(i+1:n, i+1:n)$ is zero. (Ignore roundoff.)

3. (10 points) Combine the last two parts to give an algorithm for determining the rank i of a matrix, and a linearly independent set of $n-i$ null vectors. (Again ignore roundoff.)