

Now we begin Chapter 5, on eigenvalues, eigenvectors, and diagonalizability. You might want to review Appendices D and E on complex numbers and polynomials.

We will have several motivating examples for eigenvalues:

Vibrations:

How do we know if this building will fall down in an earthquake?

How does my cell-phone "tune in" to the signal?

Data analysis:

How does Google decide the order in which to show pages?

We'll start with vibrations, using a simple example of a mass hanging from a spring, and work up to a whole building (which we'll do in less detail!). Suppose the mass is hanging at rest. Push it up, hold it steady for a moment, and let go. ASK&WAIT: What happens?

Let's write down an equation for the motion of the spring.

Our starting point is  $F = ma$  from freshman physics.

Let  $m$  = mass of the spring

$x(t)$  = distance of mass from resting position (positive means up) at time  $t$ .

For the force from the spring when it is compressed or stretched, we use  $F = -k*x$ , where  $k > 0$  is called the "spring constant".

This just means that the spring pushes down (pulls up) if the mass moves up (down), and does so with a force proportional to how far the mass has moved. So  $F = ma$  turns into

$$-k*x(t) = m*d^2 x(t)/dt^2$$

or

$$d^2 x(t)/dt^2 = -k/m * x(t)$$

Since we expect a periodic motion up and down, let's just try one of the simplest periodic functions we know,  $x(t) = \cos(w*t)$ , and plug it in to see if it works:

$$-w^2 * \cos(w*t) = -k/m * \cos(w*t)$$

which is satisfied if  $w = \sqrt{k/m}$ .  $w$  is called the "frequency of vibration" or "resonant frequency".

At  $t=0$ ,  $x(t) = 1$  and  $d x(t)/dt = w*\sin(w*t) = 0$ ,

so  $x(t) = \cos(w*t)$  satisfies the initial conditions

(mass starts with 0 velocity, pushed up distance 1 from the resting position)

Now suppose we take the top of the spring and move it up and down

at some frequency  $f$ ; this is our simplest model of an earthquake. What happens to the mass? The equation changes to

$$d^2 x(t)/dt^2 = -k/m * x(t) + \sin(f*t)$$

where  $\sin(f*t)$  is the periodic motion of the earthquake.

Suppose we start at rest (at  $t=0$ ,  $x(t) = 0$  and  $dx(t)/dt = 0$ );

What is the solution? There are two cases, depending on whether  $f=w$  (assume both  $f$  and  $w$  are positive, for simplicity)

If  $f \neq w$ , we can confirm that the solution is the sum of two sine waves, at frequencies  $f$  and  $w$ :

$$x(t) = c_1 * \sin(w*t) + c_2 * \sin(f*t) \\ \text{where } c_1 = -(f/w)/(w^2-f^2) \quad \text{and } c_2 = 1/(w^2-f^2)$$

(Homework: plug in and confirm!)

Since the sines are bounded by 1 in absolute value,

$|x(t)|$  is bounded by  $|c_1| + |c_2| = 1/|w*(w-f)|$  for all  $t$ .

So the motion  $x(t)$  is bounded for all  $t$ ,

but something clearly happens as  $f$  get closer to  $w$ :

the bound on  $|x(t)|$  gets larger and larger.

If  $f = w$ , we can confirm that the solution is

$$x(t) = c * (1 - w*t) * \sin(w*t) \\ \text{where } c = 1/(2*w^2)$$

(Homework: plug in and confirm!)

Now,  $x(t)$  is not bounded as  $t$  grows: the size of the vibrations grow proportionally to  $t$ .

ASK&WAIT: What does this mean, physically?

Physically, we say the mass-spring system is in resonance.

It would clearly not be comfortable to be riding on top of the mass when this happened, which should make you think of what might happen in an earthquake.

Now for some linear algebra: Instead of a single mass and spring, lets try hanging 3 vertically (picture), mass 3 at the bottom, mass 2 in the middle, and mass 1 at the top, with three springs (wth spring 1 at the top, spring 2 in the middle, spring 3 at the bottom). For simplicity each mass will be equal, but spring  $i$  will have spring constant  $k_i$ , and  $x_i$  will be distance of mass  $i$

from its resting position (up is positive).

Now we just write down  $F=ma$  three times, once for each mass:

$$\begin{aligned}
 m \cdot d^2 x_1(t)/t^2 &= \\
 & -k_1 * x_1 \\
 & \quad \dots \text{force from spring 1, which is compressed a distance } x_1 \\
 & -k_2 * (x_1 - x_2) \\
 & \quad \dots \text{force from spring 2, which is stretched by } x_1 - x_2 \\
 m \cdot d^2 x_2(t)/t^2 &= \\
 & -k_2 * (x_2 - x_1) \\
 & \quad \dots \text{force from spring 2, which is compressed by } x_2 - x_1 \\
 & -k_3 * (x_2 - x_3) \\
 & \quad \dots \text{force from spring 3, which is stretched by } x_2 - x_3 \\
 m \cdot d^2 x_3(t)/t^2 &= \\
 & -k_3 * (x_3 - x_2) \\
 & \quad \dots \text{force from spring 3, which is compressed by } x_3 - x_2
 \end{aligned}$$

We write these 3 equations as a single matrix equation, by letting

$$\begin{aligned}
 x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} & d^2 x(t)/dt^2 &= \begin{bmatrix} d^2 x_1(t)/dt^2 \\ d^2 x_2(t)/dt^2 \\ d^2 x_3(t)/dt^2 \end{bmatrix}
 \end{aligned}$$

$$K = -1/m * \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

and so

$$d^2 x(t)/dt^2 = K * x(t).$$

As above, we try to find a solution of the form  $x(t) = \cos(w*t) * y$ , where now  $y$  is some nonzero constant vector and  $w$  is some scalar; we need to determine both  $w$  and  $y$ . Plug in to get

$$d^2 x(t)/dt^2 = -w^2 * \cos(w*t) * y = K * x(t) = \cos(w*t) * K * y$$

The only way this can be true for all  $t$  is for

$$-w^2 * y = K * y$$

Def 1: Let  $A$  be an  $n$  by  $n$  matrix with entries from field  $F$ . Suppose there is a scalar  $\lambda$  and a nonzero vector  $y$ , all with entries from  $F$ , such that  $K * y = \lambda * y$ . Then  $\lambda$  is called an eigenvalue and  $y$  is called an

eigenvector of  $K$ . We sometimes also say  $(\lambda, y)$  is an eigenpair, or "y is  $\lambda$ 's eigenvector" or " $\lambda$  is y's eigenvalue".

Def 2: Let  $T: V \rightarrow V$  be a linear operator over a field  $F$ . Then if there is a scalar  $\lambda$  in  $F$  and a nonzero vector  $y$  in  $V$  such that  $T(y) = \lambda y$ ,  $\lambda$  is called an eigenvalue and  $y$  is an eigenvector. (This definition applies even when  $V$  is infinite dimensional.)

Note: Any nonzero multiple of an eigenvector is also an eigenvector.

Recall:  $\det(A) = 0$  iff  $A$  is noninvertible iff  $A$  has a nonzero nullspace

Thm 1:  $\lambda$  is an eigenvalue of  $A$  if and only if  $\det(A - \lambda I) = 0$ .

Proof:  $Ay = \lambda y$  iff  $(A - \lambda I)y = 0$ , i.e. iff  $A - \lambda I$  has nonzero nullspace.

Ex: 2x2 case:  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  each entry from a field  $F$

$$\begin{aligned} 0 &= \det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) \end{aligned}$$

a quadratic equation in  $\lambda$ . What are its roots?

Suppose  $F = \text{reals}$  and  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ , then

$\det(A - \lambda I) = \lambda^2 - 2\lambda$ ; there are two real roots, 0 and 2

ASK&WAIT: What are the corresponding eigenvectors?

Are they linearly independent?

Suppose  $F = \text{reals}$  and  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , then

$\det(A - \lambda I) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ ;  
there is one real root, 1, with "multiplicity 2"

What are the eigenvectors? They must satisfy  $Ay = \lambda y$ , or  $y_1 + y_2 = y_1$  and  $y_2 = y_2$ ; The first equation implies  $y_2 = 0$ ,  $y_1$  can be any nonzero scalar. In other words all eigenvectors are of the form  $\begin{bmatrix} y_1 \\ 0 \end{bmatrix}$ , i.e. multiples of  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$   
Thus there is only 1 independent eigenvector

Suppose  $F = \text{reals}$  and  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ , then

$\det(A - \lambda I) = \lambda^2 + 1$ ; there are no real roots,

but complex ones (with complex eigenvectors)

Suppose  $F = \text{rationals}$  and  $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ , then

$\det(A - \lambda I) = \lambda^2 - 2$ ; there are no rational roots,  
but real ones (with real eigenvectors)

Returning to our vibration example, we note that all numbers must be real for this to make physical sense. The solution was

$x(t) = \cos(\omega t) * y$  where  $K*y = -\omega^2*y$ . Thus  $-\omega^2$  is an eigenvalue and  $y$  is an eigenvector of  $K$ . In other words we need that the eigenvalue  $-\omega^2$  is nonpositive, in order for  $\omega$  to be real. This is indeed the case for this  $K$ .

Suppose now that we "shake" the masses according to the following equation, much as we shook the single mass:

$$d^2 x(t)/dt^2 = K*x + \sin(f*t)*y$$

Let's try to find a solution of the form  $x(t) = z(t)*y$ , where  $z(t)$  is a scalar function:

$$\begin{aligned} d^2 x(t)/dt^2 &= d^2 z(t)/dt^2 * y \\ &= K*z(t)*y + \sin(f*t)*y \\ &= z(t)*(-\omega^2)*y + \sin(f*t)*y \\ &= (-\omega^2*z(t) + \sin(f*t)) * y \end{aligned}$$

For this to be true, we must have

$$d^2 z(t)/dt^2 = -\omega^2 * z(t) + \sin(f*t)$$

which is exactly the same scalar equation we had before, with bounded solutions when  $\omega \neq f$ , and "resonance" (unbounded solutions) when  $\omega = f$ , i.e. when the eigenvalue  $-\omega^2 = -f^2$ .

Now what about deciding if a building will fall down in an earthquake? This is a complicated question, but here is one approach. The structural parts of a building, such as the steel beams, may be modeled as masses and springs, of course connected in three dimensions. Writing down  $F = ma$  for all the pieces of the building (a process called "finite element analysis" in civil and mechanical engineering) again leads to an equation

$$d^2 x(t)/dt^2 = K * x(t)$$

where  $K$  is typically enormous, with its dimension  $n$  usually many thousands or even sometimes millions. Again we get solutions of the form  $x(t) = \cos(\omega t)*y$ , where  $-\omega^2$  is an eigenvalue of  $K$  and  $y$  is an eigenvector. Now  $K$  will have  $n$  eigenpairs  $(\lambda_1, y_1), (\lambda_2, y_2), \dots, (\lambda_n, y_n)$ ,

corresponding to the roots of its characteristic polynomial of degree  $n$ . If the building is shaken, say by an earthquake, at a frequency  $f$  that matches one of these eigenvalues ( $\lambda_i \sim -f^2$  for some  $i$ ), then the building will undergo large vibrations, and be in danger of falling down. So what an engineer does is look at seismographic recordings of known earthquakes, and see what the usual frequencies of earthquakes are. (They are typically between 1 and 10 vibrations per second) If any of these match your buildings eigenvalues, you're in trouble. Hopefully the design engineer will discover this while designing the building and change the design before the building is built.

There are several famous bridges that have started vibrating because of this resonance. The Tacoma Narrows bridge collapsed in 1940 because of resonance caused by wind rather than earthquakes. In the on-line video at [www.enm.bris.ac.uk/research/nonlinear/tacoma/tacoma.html](http://www.enm.bris.ac.uk/research/nonlinear/tacoma/tacoma.html) the sine-wave oscillations are clearly evident.

The Millennium Bridge for pedestrians, which opened over the River Thames in London in 2000, was closed shortly thereafter because the pedestrian traffic set off a resonance that required a redesign and retrofit. The web site [www.arup.com/MillenniumBridge](http://www.arup.com/MillenniumBridge) includes a video of the vibrations. An analysis of this phenomenon was recently published by Steve Strogatz of Cornell in Nature (3 Nov 2005 issue, [www.nature.com](http://www.nature.com)).

We are currently engaged in a project to put motion sensors onto the Golden Gate Bridge to detect onset of such vibrations caused by wind (which will not cause the GGB to collapse, but to possibly move several 10s of feet east to west, causing serious traffic problems).

Resonance is not always bad, as in these examples. Indeed, all radio, TV and other telecommunications depends on it, because it is how tiny electrical signals are detected and amplified. For example, in your cell phone the tiny electrical signal in the antenna causes a quartz crystal to resonate mechanically, which is how the signal is detected. There is so little energy, and so much damping ("friction" in the material) that the vibrations remain microscopic, and indeed

cell-phone designers would like to make them larger, because this would save (battery) power, not just in cell-phones but in all telecommunication devices. A number of research groups are engaged in trying to design new resonators that resonate as strongly as possible at cell-phone frequencies (gigaHertz), which amounts to adjusting their shape to make their eigenvalues have certain target values.

We will consider the Google example after we have more background.

Now we go back to the formal treatment of eigenvalues and eigenvectors.

Thm 2:  $\det(A - \lambda I)$  is a polynomial in  $\lambda$  of degree  $n$ . (This polynomial is called the characteristic polynomial of  $A$ ).

Thus the roots of the characteristic polynomial (those which are in the field  $F$  of entries of  $A$ , the only scalars we are formally allowed to talk about) are the eigenvalues.

Ex: Suppose  $A$  is diagonal. Then  $A - \lambda I$  is also diagonal and  $\det(A - \lambda I) = \prod_{i=1}^n (A_{ii} - \lambda)$ , which is a polynomial with roots  $A_{11}, \dots, A_{nn}$ , so the eigenvalues are just the diagonal entries.

ASK&WAIT: What if  $A$  is upper triangular?

Proof 1: One of our equivalent definitions of a determinant, the "explicit formula", was to multiply out the determinant explicitly, getting a big polynomial in the matrix entries, where we multiply  $n$  entries together in each term of the polynomial. Since each entry of  $A - \lambda I$  is either  $a_{ij}$  or  $a_{ii} - \lambda$ , multiplying  $n$  such factors together gives us terms of degree at most  $n$ . Multiplying all the diagonal entries  $a_{ii} - \lambda$  gives us a term  $(-1)^n * \lambda^n$ , the only one of degree  $n$  in  $\lambda$ .

Proof 2: We can expand along the first row and use the induction hypothesis that all the  $n-1$  by  $n-1$  determinants are polynomials of degree at most  $n-1$ . (Homework!)

The next goal is to show that any polynomial can be factored into a product of factors of the form  $(x - r_i)$  using its roots  $r_i$ .

Some basic facts about polynomials are reviewed in Appendix E.

Lemma 1: If  $r$  is a root of polynomial  $p(x)$ , i.e.  $p(r) = 0$ , then we can write  $p(x) = (x-r)q(x)$ , where  $q(x)$  is a polynomial of one lower degree than  $p(x)$ .

Proof: Just divide  $p(x)$  by  $(x-r)$  getting

$$p(x) = (x-r)q(x) + \text{remainder}$$

and note that the remainder is a constant that must be 0.

See appendix E.

Def 3: A polynomial  $p(x)$  has a root  $r$  of multiplicity  $k$  if we can write  $p(x) = (x-r)^k * q(x)$  where  $q(x)$  is another polynomial, and  $k$  is the largest integer for which this is true.

Lemma 2: If  $p(x)$  has degree  $n$  and  $q(x)$  has degree  $m$  then  $p(x)q(x)$  has degree  $n+m$

Proof: Exercise!

Thm 3: An  $n$  by  $n$  matrix can have at most  $n$  eigenvalues, counting multiplicities. In other words, if  $r_1$  through  $r_k$  are distinct roots of  $p(x)$ , where  $r_i$  has multiplicity  $m_i$ , then  $m_1 + m_2 + \dots + m_k \leq n$ .

Proof of Thm 3: Let  $p(x) = \det(A - \lambda I)$  be the characteristic polynomial. By assumption  $(x-r_1)^{m_1}$  through  $(x-r_k)^{m_k}$  all divide  $p(x)$ . We claim that we can therefore factor  $p(x) = (x-r_1)^{m_1} * (x-r_2)^{m_2} * \dots * (x-r_k)^{m_k} * s(x)$ , where  $s(x)$  is another polynomial. Assuming for a moment that this is true, then by Lemma 2 the degree of  $p$  is the sum of the degrees of its factors, and so  $\text{degree}(p) = n = m_1 + m_2 + \dots + m_k + \text{degree}(s) \geq m_1 + \dots + m_k$  as desired.

To prove that  $p(x)$  factors as described, we can use induction.

on the  $m_i$ . We start writing  $p(x) = (x-r_1)^{m_1} * q(x)$  which we are given. Since  $(x-r_2)^{m_2}$  also divides  $p(x)$ , we get  $p(r_2)=0$ , and so  $q(r_2)=0$  (since  $(r_2 - r_1) \neq 0$ ).

So by Lemma 1,  $q(x) = (x-r_2)q_2(x)$ .

Thus  $p(x) = (x-r_1)^{m_1} * (x-r_2)q_2(x)$ . Since  $(x-r_2)^{m_2}$  divides  $p(x)$ , we get  $(x-r_2)^{(m_2-1)}$  divides  $(x-r_1)^{m_1}q_2(x)$ , so  $q_2(r_2)=0$  and

we can factor out another  $(x-r_2)$ . We keep repeating this for each factor of the form  $(x-r_i)$ .

Corollary 1: The number of distinct eigenvalues of an  $n$  by  $n$  matrix  $A$  is at most  $n$ .

Proof: If  $r_1, \dots, r_k$  are distinct roots, Thm 3 implies  $n \geq m_1 + \dots + m_k \geq k$ .

As pointed out above, not all polynomials have  $n$  roots, counting multiplicities, if we insist the roots have to be in the same field as the polynomial coefficients. For example, polynomials with rational entries may have irrational or complex roots.

Def 4: A polynomial with coefficients over  $F$  "splits" if all its roots are in  $F$ .

Ex:  $x^2 - 2$  splits over  $R$  but not  $Q$ .  $x^2 + 1$  splits over  $C$  but not  $R$ .

Def 5: A field  $F$  with the property that all polynomials with coefficients from  $F$  have all roots in  $F$  is called algebraically closed.

The field of complex numbers is algebraically closed, as shown in Appendix D. In Ma114 you may study other algebraically closed fields. In some of our results we will assume  $F$  is algebraically closed, in others we will not. We will always assume that our fields are infinite (i.e. no finite fields like  $Z_2$ ).

We briefly consider eigenvalues of linear operators  $T:V \rightarrow V$  for infinite dimension  $T$ . One can imagine that one no longer necessarily gets a finite set of eigenvalues. Here is an example:

Let  $V = \{\text{"smooth" functions from } R \rightarrow R, \text{ i.e. they have derivatives of all orders}\}$   
Let  $T(f) = f'$ , i.e. taking the derivative. Then  $T(f) = \lambda f$  means  $f'(x) = \lambda f(x)$ , a differential equation with solution  $f(x) = c \exp(\lambda x)$ . Thus every real number is an eigenvalue of  $T$ .

Recall that many results of Chapter 3 (and even Chapter 4) came down to understanding and using a single "matrix decomposition", namely the LU decomposition. There is a similar matrix decomposition for

Chapter 5, which we now discuss. Unlike the LU decomposition, it does not exist for every matrix (for reasons including the fact that all roots of a polynomial may not exist in the field  $F$ ). But it exists for "most" matrices over  $C$ , and is very useful. We will call it an eigendecomposition (the book also says "diagonalization").

Def 6: An eigendecomposition of an  $n$  by  $n$  matrix  $A$  is the expression  $A = Q * \text{Lambda} * Q^{-1}$  where  $Q$  is an invertible matrix, and  $\text{Lambda} = \text{diag}(\text{lambda}_1, \dots, \text{lambda}_n)$  is diagonal. If this decomposition exists, we say  $A$  is diagonalizable.

Recall the definition of two matrices being similar:  $A = Q*B*Q^{-1}$  means  $A$  and  $B$  are similar. So another way to define diagonalizability is that  $A$  is diagonalizable if it is similar to a diagonal matrix.

Thm 4: An eigendecomposition  $A = Q * \text{Lambda} * Q^{-1}$  exists if and only if

- (1) for all  $i$ ,  $\text{lambda}_i$  is an eigenvalue of  $A$
- (2) column  $i$  of  $Q$  (call it  $q_i$ ) is the eigenvector of  $A$  belonging to  $\text{lambda}_i$
- (3) The  $n$  eigenvectors  $\{q_1, \dots, q_n\}$  are linearly independent

Proof: Assume the eigendecomposition exists.

Multiply  $A = Q*\text{Lambda}*Q^{-1}$  on the right by  $Q$  to get the equivalent equality  $A*Q = Q * \text{Lambda}$ . Multiply this by the  $j$ -th standard basis vector  $e_j$  to get

$$\begin{aligned}
 \text{column } j \text{ of } A*Q &= (A*Q)*e_j = (Q*\text{Lambda})*e_j = \text{column } j \text{ of } Q*\text{Lambda} \\
 (A*Q)*e_j &= A*(Q*e_j) \quad \dots \text{ by associativity} \\
 &= A*q_j \quad \dots \text{ where } q_j = \text{column } j \text{ of } Q \\
 &= (Q * \text{Lambda})*e_j \\
 &= Q * (\text{Lambda}*e_j) \quad \dots \text{ by associativity} \\
 &= Q * (\text{lambda}_j*e_j) \quad \dots \text{ since } \text{Lambda} \text{ is diagonal} \\
 &= \text{lambda}_j * (Q * e_j) \quad \dots \text{ factoring out } \text{lambda}_j \\
 &= \text{lambda}_j * q_j
 \end{aligned}$$

or  $A*q_j = \text{lambda}_j * q_j$ , i.e. for all  $j$ ,  $\text{lambda}_j$  is an eigenvalue and  $q_j$  is its eigenvector. The fact that  $Q = [q_1, q_2, \dots, q_n]$  is assumed to be invertible means that its columns are linearly independent (otherwise  $Q$  would have a nonzero nullspace).

Conversely, if (1) and (2) hold, the above argument shows that  $A*q_j = \text{lambda}_j * q_j$  can be written as  $A*Q = Q*\text{Lambda}$ .

Now (3) implies that  $Q$  is invertible, and so we can multiply  $A*Q = Q*\Lambda$  on the right by  $Q^{-1}$  to get  $A*Q*Q^{-1} = A = Q*\Lambda*Q^{-1}$ , the desired eigendecomposition.

It is easy to compute the characteristic polynomial of a diagonalizable matrix:

$$\begin{aligned}
 \det(A - x*I) &= \det(Q*\Lambda*Q^{-1} - x*I) \\
 &= \det(Q*\Lambda*Q^{-1} - x*Q*I*Q^{-1}) \\
 &\quad \dots \text{ since } Q*I*Q^{-1}=I \\
 &= \det(Q*(\Lambda - x*I)*Q^{-1}) \\
 &\quad \dots \text{ factoring out } Q \text{ and } Q^{-1} \\
 &= \det(Q) * \det(\Lambda - x*I) * \det(Q^{-1}) \\
 &\quad \dots \text{ since } \det(A*B) = \det(A)*\det(B) \\
 &= \det(Q) * \det(Q^{-1}) * \det(\Lambda - x*I) \\
 &= \det(\Lambda - x*I) \\
 &\quad \dots \text{ since } \det(Q) * \det(Q^{-1}) = 1 \\
 &= \text{product}_{\{i=1 \text{ to } n\}} (\lambda_i - x) \\
 &\quad \dots \text{ since } \Lambda - x*I \text{ is diagonal}
 \end{aligned}$$

whose roots are evidently  $\lambda_1$  through  $\lambda_n$ , as expected.

Now we go back to our first motivating problem, solving a system of linear differential equations:

$$d x(t)/dt = A * x(t)$$

where  $x(t)$  is a vector of unknown functions,  $A$  is an  $n$  by  $n$  matrix, and  $f(t)$  is a vector of known functions. Suppose we also know  $x(0)$ .

If  $A$  is diagonalizable, this is very easy to solve: Write

$$d x(t)/dt = Q*\Lambda*Q^{-1} * x(t)$$

and multiply through by  $Q^{-1}$  to get

$$Q^{-1} * d x(t)/dt = \Lambda * Q^{-1} * x(t)$$

Define a new vector of unknowns  $y(t) = Q^{-1} * x(t)$ . Then

$$y(0) = Q^{-1} * x(0) \text{ and}$$

$$d y(t)/dt = \Lambda * y(t)$$

Since  $\Lambda$  is diagonal we can write this as  $n$  separate equations:

$$d y_1(t)/dt = \lambda_1 * y_1(t)$$

$$d y_2(t)/dt = \lambda_2 * y_2(t)$$

...

$$d y_n(t)/dt = \lambda_n * y_n(t)$$

each of which is easy to solve:  $y_i(t) = y_i(0) * \exp(\lambda_i * t)$

Given each component of  $y(t)$ , we can therefore get the solution of the original equation

$$x(t) = Q * y(t)$$

$$\begin{aligned}
&= Q * \begin{bmatrix} y_1(0) * \exp(\lambda_1 * t) \\ \dots \\ y_n(0) * \exp(\lambda_n * t) \end{bmatrix} \\
&= Q * \begin{bmatrix} \exp(\lambda_1 * t) & 0 & 0 \\ 0 & \exp(\lambda_2 * t) & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \exp(\lambda_n * t) \end{bmatrix} * \begin{bmatrix} y_1(0) \\ y_2(0) \\ \dots \\ y_n(0) \end{bmatrix} \\
&= Q * \begin{bmatrix} \exp(\lambda_1 * t) & 0 & 0 \\ 0 & \exp(\lambda_2 * t) & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \exp(\lambda_n * t) \end{bmatrix} * Q^{-1} * x(0)
\end{aligned}$$

To solve the more general system  $\frac{dx(t)}{dt} = A * x + f(t)$ , where  $f(t)$  is a vector function, the same idea also reduces it to solving  $n$  scalar differential equations, which we know how to do from Ma53:

$$\begin{aligned}
\frac{dx(t)}{dt} &= A * x(t) + f(t) \\
&= Q * \Lambda * Q^{-1} * x(t) + f(t)
\end{aligned}$$

Multiply through by  $Q^{-1}$  to get

$$Q^{-1} * \frac{dx(t)}{dt} = \Lambda * Q^{-1} * x(t) + Q^{-1} * f(t)$$

Define a new vector of unknowns  $y(t) = Q^{-1} * x(t)$  and

known vector  $g(t) = Q^{-1} * f(t)$ , so  $y(0) = Q^{-1} * x(0)$  and

$$\frac{dy(t)}{dt} = \Lambda * y(t) + g(t)$$

Since  $\Lambda$  is diagonal we can write this as  $n$  separate equations:

$$\frac{dy_1(t)}{dt} = \lambda_1 * y_1(t) + g_1(t)$$

...

$$\frac{dy_n(t)}{dt} = \lambda_n * y_n(t) + g_n(t)$$

each of which we know how to solve from Ma53.

Given each component of  $y(t)$ , we can therefore get the solution of the original equation via  $x(t) = Q * y(t)$ .

We have already seen that some matrices are not diagonalizable.

For example, we saw that  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  had only one eigenvector, whereas we would need two independent eigenvectors to diagonalize it.

Eventually (Chapter 7) we will see how to generalize eigendecompositions to this case (using the Schur decomposition or Jordan Canonical Form), but in the meantime let us state some simple theorems about when a matrix is diagonalizable. (We will have more such theorems later.)

Thm 5: If  $\lambda_1, \dots, \lambda_k$  are distinct eigenvalues of  $A$ , then their eigenvectors  $q_1, \dots, q_k$  are linearly independent.

Corollary 2: If all the eigenvalues of  $A$  are distinct, then  $A$  is diagonalizable.

Proof of Corollary 2: If the  $n$  eigenvectors  $q_1, \dots, q_n$  are independent, then the matrix  $Q = [q_1, \dots, q_n]$  is nonsingular, and so by Thm 4,  $A$  is diagonalizable.

Proof of Thm 5: We use proof by contradiction.

Suppose  $q_1, \dots, q_k$  are dependent, that is

$$\sum_{i=1}^k a_i q_i = 0$$

for some  $a_i$ , not all zero. Assume without loss of generality that  $a_k \neq 0$ . Now multiply the above equation by  $A - \lambda_1 I$ :

$$\begin{aligned} 0 &= (A - \lambda_1 I) * \sum_{i=1}^k a_i q_i \\ &= \sum_{i=1}^k a_i * (A - \lambda_1 I) q_i \\ &\quad \dots \text{ since } A - \lambda_1 I \text{ is a linear transformation} \\ &= \sum_{i=1}^k a_i * (\lambda_i - \lambda_1) q_i \\ &\quad \dots \text{ since } A q_i = \lambda_i q_i \\ &= \sum_{i=2}^k a_i * (\lambda_i - \lambda_1) q_i \\ &\quad \dots \text{ since the } i=1 \text{ term is zero} \\ &= \sum_{i=2}^k a'_i q_i \\ &\quad \dots \text{ where } a'_i = a_i * (\lambda_i - \lambda_1) \end{aligned}$$

where each  $a'_i$  is nonzero iff  $a_i$  is nonzero, since  $(\lambda_i - \lambda_1)$  is nonzero for  $i > 1$ .

Now take the last equation and multiply it by  $A - \lambda_2 I$ .

The same logic shows

$$0 = \sum_{i=3}^k a''_i q_i$$

where each  $a''_i = (\lambda_i - \lambda_2) * (\lambda_i - \lambda_1) * a_i$  is nonzero iff  $a_i$  is nonzero. Continue multiplying by

$A - \lambda_3 I, A - \lambda_4 I, \dots, A - \lambda_{k-1} I,$   
until we get

$$0 = (\lambda_k - \lambda_1) * \dots * (\lambda_k - \lambda_{k-1}) * a_k q_k$$

All the factors  $(\lambda_k - \lambda_i)$  are nonzero because the all the  $\lambda_i$  are distinct. Thus  $a_k = 0$ , but this contradicts the assumption that  $a_k \neq 0$ .

A matrix can be diagonalizable even when the eigenvalues are not distinct. ASK&WAIT: Can you give a simple example? Where all the eigenvalues are identical?

Here is a more general theorem about diagonalizability that takes multiple eigenvalues into account.

Def 7: Let  $A$  be a matrix with eigenvalue  $\lambda$ .  
 Then  $E_\lambda = \text{Null}(A - \lambda I) = \{x: A*x = \lambda*x\}$   
 is called the eigenspace of  $T$  corresponding to  $\lambda$ .

Note that if  $x$  in  $E_\lambda$ , then  $A*x = \lambda*x$ .  
 ASK&WAIT: Does this means  $E_\lambda = \{\text{all eigenvectors of } A \text{ for } \lambda\}$ ?

Ex: If  $A = I$ , with eigenvalue 1, then  $E_1 = \text{Null}(I - 1*I) = F^n$ ,  
 all vectors.

Ex: If  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$  then  $E_2 = \text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$  and  $E_3 = \text{span}\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$

Thm 6: If  $\lambda$  is an eigenvalue of  $A$  with multiplicity  $m$ ,  
 then  $\dim(E_\lambda) \leq m$ . In other words, there are at most  
 $m$  linearly independent eigenvectors for eigenvalue  $\lambda$ .

Proof: Let  $\{v_1, \dots, v_k\}$  be an ordered basis for  $E_\lambda$ .  
 We want to show  $k \leq m$ . Extend  $\{v_1, \dots, v_k\}$  to an ordered basis  
 of  $F^n$ :  $\{v_1, \dots, v_n\}$ . Let  $Q = [v_1, \dots, v_n]$ ,  
 which must be a nonsingular matrix.

Then the  $j$ -th column of  $A*Q$  is  
 $A*Q*e_j = A*(Q*e_j) = A*v_j = \lambda*v_j$  if  $j \leq k$ .  
 In other words,  $A*Q = [v_1, \dots, v_k, x_{k+1}, \dots, x_n]$   
 where  $x_{k+1}, \dots, x_n$  are some other vectors that we  
 don't need to know more about. Then when  $j \leq k$ ,  
 the  $j$ -th column of  $Q^{-1}*A*Q$  is

$$\begin{aligned} (Q^{-1}*A*Q)*e_j &= Q^{-1}*(A*Q*e_j) \\ &= Q^{-1}*(\lambda*v_j) \\ &= \lambda * Q^{-1}*(v_j) \\ &= \lambda * Q^{-1}*(Q*e_j) \\ &= \lambda * (Q^{-1}*Q)*e_j \\ &= \lambda * (I)*e_j \\ &= \lambda * e_j \end{aligned}$$

In other words

$$Q^{-1}*A*Q = \begin{bmatrix} \lambda I & B \\ 0 & C \end{bmatrix} \begin{matrix} k & n-k \\ k & n-k \end{matrix}$$

where  $B$  and  $C$  are matrices we don't need to know more about.  
 Thus the characteristic polynomial  $p(x)$  of  $A$  is

$$\begin{aligned}
\det(A - xI) &= \det(Q^{-1}) \det(A - xI) \det(Q) \\
&= \det(Q^{-1} (A - xI) Q) \\
&= \det\left( \begin{bmatrix} (\lambda - x)I & B \\ 0 & C - xI \end{bmatrix} \right) \\
&= \det((\lambda - x)I) \det(C - xI) \\
&= (\lambda - x)^k \det(C - xI)
\end{aligned}$$

so  $(\lambda - x)^k$  is a factor of  $p(x)$ . Thus, by the definition of multiplicity,  $k \leq m$ , as desired.

Thm 7.  $A$  is diagonalizable if and only if

$$n = \sum_{i=1}^k \dim(E_{\lambda_i})$$

where  $\lambda_1, \dots, \lambda_k$  are all the distinct eigenvalues of  $A$ .

Proof: First, suppose that  $A$  is diagonalizable:

$$A = Q \Lambda Q^{-1}$$

where we have numbered the eigenvalues and eigenvectors (columns of  $Q$ ) so that

the first  $n_1$  eigenvalues are equal with common value  $\lambda_1$ ,

the next  $n_2$  eigenvalues are equal with common value  $\lambda_2 \neq \lambda_1$ ,

and so on. Thus the first  $n_1$  columns of  $Q$ , call them

$$q^{(1)}_1, \dots, q^{(1)}_{n_1}$$

are  $n_1$  independent eigenvectors for  $\lambda_1$ ,

the next  $n_2$  columns of  $Q$ , call them

$$q^{(2)}_1, \dots, q^{(2)}_{n_2}$$

are  $n_2$  independent eigenvectors for  $\lambda_2$ , and so on. Thus

$$E_{\lambda_i} = \text{Null}(A - \lambda_i I) = \text{span}(q^{(i)}_1, \dots, q^{(i)}_{n_i})$$

and

$$\dim(E_{\lambda_i}) = n_i.$$

and

$$\sum_{i=1}^k \dim(E_{\lambda_i}) = \sum_{i=1}^k n_i = n$$

as desired.

Second, suppose  $n = \sum_{i=1}^k \dim(E_{\lambda_i})$

Define  $n_i = \dim(E_{\lambda_i})$ . Let

$$Q_i = [q^{(i)}_1, \dots, q^{(i)}_{n_i}]$$

be an  $n$  by  $n_i$  matrix whose columns are an ordered basis for  $E_{\lambda_i}$ .

Let

$$Q = [Q_1, \dots, Q_k]$$

be an  $n$  by  $n$  matrix. Then by construction

$$A Q = Q \Lambda$$

where  $\Lambda$  is a diagonal matrix whose first  $n_1$  entries are

$\lambda_1$ , next  $n_2$  entries of  $\lambda_2$ , and so on.

To prove  $A$  is diagonalizable, By Thm 4 all we still need to show is that  $Q$  is invertible, i.e. its columns are independent, i.e. if some linear combination

$$0 = \sum_{i=1}^k \sum_{j=1}^{n_i} a_{ij} q^{(i)}_j$$

then we need to show all the  $a_{ij} = 0$ . Note that each inner sum

$$z_i = \sum_{j=1}^{n_i} a_{ij} q^{(i)}_j$$

gives a vector  $z_i$  inside the subspace  $E_{\{\lambda_i\}}$ , i.e. an eigenvector for  $\lambda_i$ . Thus

$$0 = \sum_{i=1}^k z_i$$

But by Thm 5, this can only be true if each  $z_i = 0$ , since (nonzero) eigenvectors corresponding to distinct eigenvalues are independent.

But by the independence of  $q^{(i)}_1, \dots, q^{(i)}_{n_i}$  in the sum for  $z_i$ , all the coefficients  $a_{ij}$  must be zero.

We use diagonalizability to talk about computing functions of matrices.

If  $A$  is a matrix,  $A^i$  means  $A \cdot A \cdot \dots \cdot A$ , ( $i$  factors of  $A$ ).

Def 8: If  $p(x) = \sum_{i=0}^d p_i x^i$  is a polynomial, then for any square matrix

$$p(A) = \sum_{i=0}^d p_i A^i.$$

Thm 8: Suppose  $A$  is diagonalizable:  $A = Q \cdot \Lambda \cdot Q^{-1}$  with  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ .

Then  $p(A) = Q \cdot p(\Lambda) \cdot Q^{-1}$

$$= Q \cdot \text{diag}(p(\lambda_1), \dots, p(\lambda_n)) \cdot Q^{-1}$$

is also diagonalizable, with eigenvalues  $p(\lambda_i)$

and the same eigenvectors as  $A$ .

Proof:

$$\begin{aligned} A^i &= (Q \cdot \Lambda \cdot Q^{-1})^i \\ &= (Q \cdot \Lambda \cdot Q^{-1}) \cdot (Q \cdot \Lambda \cdot Q^{-1}) \cdot \dots \cdot (Q \cdot \Lambda \cdot Q^{-1}) \\ &\quad \dots \text{ i copies of } (Q \cdot \Lambda \cdot Q^{-1}) \\ &= Q \cdot \Lambda \cdot (Q^{-1} \cdot Q) \cdot \Lambda \cdot (Q^{-1} \cdot Q) \cdot \dots \cdot (Q^{-1} \cdot Q) \cdot \Lambda \cdot Q^{-1} \\ &\quad \dots \text{ associativity} \\ &= Q \cdot \Lambda \cdot (I) \cdot \Lambda \cdot (I) \cdot \dots \cdot (I) \cdot \Lambda \cdot Q^{-1} \\ &\quad \dots \text{ "cancelling" each } Q^{-1} \cdot Q \\ &= Q \cdot \Lambda^i \cdot Q^{-1} \end{aligned}$$

so

$$\begin{aligned} p(A) &= \sum_{i=0}^d p_i A^i \\ &= \sum_{i=0}^d p_i Q \cdot \Lambda^i \cdot Q^{-1} \end{aligned}$$

$$\begin{aligned}
&= Q * (\sum_{i=0 \text{ to } d} p_i * \text{Lambda}^i ) * Q^{-1} \\
&= Q * p(\text{Lambda}) * Q^{-1}
\end{aligned}$$

Furthermore

$$\begin{aligned}
p(\text{Lambda}) &= \sum_{i=0 \text{ to } d} p_i * \text{Lambda}^i \\
&= \sum_{i=0 \text{ to } d} p_i * \text{diag}(\text{lambda}_1^i , \dots , \text{lambda}_n^i ) \\
&= \text{diag}( \sum_{i=0 \text{ to } d} p_i * \text{lambda}_1^i , \dots , \\
&\quad \sum_{i=0 \text{ to } d} p_i * \text{lambda}_n^i ) \\
&= \text{diag}( p(\text{lambda}_1) , \dots , p(\text{lambda}_n) )
\end{aligned}$$

Corollary 3: (Cayley-Hamilton Thm for diagonalizable matrices)

Let  $p(x)$  be the characteristic polynomial of  $A = Q * \text{Lambda} * Q^{-1}$ . Then  $p(A) = 0$ , the zero matrix.

Proof: Thm 8 implies

$$\begin{aligned}
p(A) &= Q * p(\text{Lambda}) * Q^{-1} \\
&= Q * \text{diag}(p(\text{lambda}_1), \dots, p(\text{lambda}_n)) * Q^{-1} \\
&= Q * \text{diag}(0, \dots, 0) * Q^{-1} \\
&\quad \dots \text{ since the eigenvalues are roots of } p(x) \\
&= 0
\end{aligned}$$

(Eventually we will prove  $p(A)=0$  for all  $A$ , even when  $A$  is not diagonalizable, which is the complete statement of the Cayley-Hamilton Thm.)

Now we consider functions defined by Taylor series, not just polynomials. We will say that a sequence of matrices  $A^{(k)}$  converges to a limit matrix  $A$  as  $k \rightarrow$  infinity, if each component  $A^{(k)}_{ij}$  converges to  $A_{ij}$ . Suppose  $f(x) = \sum_{i=0 \text{ to } \infty} f_i * x^i$  is a Taylor series expansion that converges in some region; this means that the limit of the following sequence of polynomials converges:

$$\lim_{n \rightarrow \infty} \sum_{i=0 \text{ to } n} f_i * x^i$$

We will say that  $f(A)$  converges to a limit if the sequence of polynomials

$$\lim_{n \rightarrow \infty} \sum_{i=0 \text{ to } n} f_i * A^i$$

converges to a matrix.

Thm 9: Suppose  $f(x) = \sum_{i=0 \text{ to } \infty} f_i * x^i$  converges for  $x$  equal to any of the eigenvalues of the diagonalizable matrix  $A = Q * \text{Lambda} * Q^{-1}$ , where  $\text{Lambda} = \text{diag}(\text{lambda}_1, \dots, \text{lambda}_n)$ .

Then  $f(A)$  converges, and is diagonalizable:

$$f(A) = Q * \text{diag}( f(\text{lambda}_1), \dots, f(\text{lambda}_n) ) * Q^{-1}$$

with eigenvalues  $f(\text{lambda}_i)$  and the same eigenvectors as  $A$ .

Proof: Let  $f_N(x) = \sum_{i=0 \text{ to } N} f_i * x^i$ . Then

from Thm 8,

$$f_N(A) = Q * \text{diag}(f_N(\lambda_1), \dots, f_N(\lambda_n)) * Q^{-1}$$

As  $N \rightarrow \infty$ , by assumption  $f_N(\lambda_i) \rightarrow f(\lambda_i)$ .

Thus

$$f_N(A) \rightarrow Q * \text{diag}(f(\lambda_1), \dots, f(\lambda_n)) * Q^{-1}$$

as desired.

Example: suppose  $\frac{dx(t)}{dt} = A*x(t)$ ,  $x(0)$  given.

with  $A = Q * \text{diag}(\lambda_1, \dots, \lambda_n) * Q^{-1}$

diagonalizable. Recall that the solution is

$$x(t) = Q * \text{diag}(\exp(\lambda_1*t), \dots, \exp(\lambda_n*t)) * Q^{-1} * x(0)$$

We can now rewrite this as

$$x(t) = \exp(t*A) * x(0)$$

by applying Thm 9 to the diagonalizable matrix

$$t*A = Q * \text{diag}(t*\lambda_1, \dots, t*\lambda_n) * Q^{-1}$$