Math 110 - Fall 05 - Lectures notes # 3 - Sep 2 (Friday)

Goal for the day: Recall def of vector space
Define subspace, give examples, basic properties

Def Let V be a vector space over field F. A subset W of V is called
a subspace of V if it is also a vector space, with the same
operations + and * used for V.

EX; V is a subspace of V, {0_V} is a subspace of V, called the zero subspace

To prove that subset W of V is a subspace, we need to confirm that
axioms VS1 through VS8 all hold. Fortunately, VS1, VS2, VS5,
VS6, VS7 and VS8 (commutativity, associativity, distributivity, mult by 1)
all hold automatically, because they hold for V.
So we only need to worry about
(1) Is W closed under addition?: for all x,y in W, x+y is in W
(2) Is W closed under scalar mult?: for all x in W and t in F, t*x is in W
(3) W contains a zero vector, called 0_W (VS3)
(4) if x is in W so is -x (VS4)

In fact, it is enough to check (1), (2) and (3) because (4)
can be shown to be true if (1) and (2) are true:

Thm 1: Let W be a subset of vector space V (over field F).
Then W is a subspace of V if and only if
(1) 0_V is in W
(2) for all x and y in W, x+y is in W
(3) for all x in W and t in F, t*x is in W

Proof: We first assume W is a subspace, and then prove (1), (2) and (3) hold.
(2) and (3) hold by the definition of subspace.
To prove (1), let 0_W be the zero vector for W; we have to show 0_W = 0_V.
But for any w in W, w + 0_W = w = w + 0_V, by the defs of 0_W and 0_V.
Applying the Cancellation Law, 0_W = 0_V

Next we assume (1), (2) and (3) hold, and show W is a subspace.
(2) and (3) imply that W is closed under + and * as required.
VS3 (existence of 0_W in W) holds for W because we can set 0_W = 0_V.
VS4 (-x is in W for all x in W) holds for W because a Theorem from
last time (Thm 1.2, section 1.2) says -x = (-1)*x, which is in W by (3)

It is reasonable to ask whether we really need part (1): Why
not just use part (3) with t = 0_F to conclude that 0_F*x = 0_V is in W?
Here is why. W = null set satisfies (2) and (3), but not (1) ((2) and
(3) are "vacuously" true). But the null set is not a vector space,
because one of the requirements of a vector space is that it
contain a zero vector.

Ex 1 of subspace: Let V = F^2 = {(x,y), x and y in F},
   and let W = {(x,0_F), x in F} = "x axis"
ASK & WAIT: why is W a subspace?

Ex 2 of subspace: Let V = F^2, choose any f in F, let W_f = {(f*x,x), x in F}
ASK & WAIT: why is W_f a subspace? What does it look like, geometrically,
            as a subset of F^2?

Ex 3 of subspace: Let V = F^3, choose any nonzero f=(f1,f2,f3) in V,
let W_f = {(f1*x,f2*x,f3*x), x in F}
ASK & WAIT: why is W_f a subspace? What does it look like, geometrically?

Ex 4 of subspace: Let V = F^3, choose nonzero f = (f1,f2,f3) in V,
Let W'_f = {(x1,x2,x3) such that x1*f1 + x2*f2 + x3*f3 = 0_F}
ASK & WAIT: Why is W'_f a subspace? What does it look like geometrically?

Ex 5 of subspace: Let V = F^3, choose nonzero f = (f1,f2,f3) and
g = (g1,g2,g3) in V, Let W'_f,g = {(x1,x2,x3) such that
    x1*f1 + x2*f2 + x3*f3 = 0_F = x1*g1 + x2*g2 + x3*g3}
ASK & WAIT: Why is W'_f,g a subspace? What does it look like geometrically?
            Note: We can also write W'_f,g = W'_f intersect W'_g

This is a special case of

Thm 2: An intersection of subspaces of V is also a subspace
Proof: Let C be a set of subspaces of V. Let W be the
intersection of all these subspaces. We need to confirm that
the 3 conditions of Thm 1 apply to W.
(1) let W_c be any of the subspaces in C. Then since 0_V is in W_c,
it is in their intersection, namely W.
(2) Let x and y be in W. Then they must also be in W_c, so
    x+y is in W_c. Since this is true for all W_c, x+y is in
    their intersection, namely W
(3) same idea as (2)

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Recall vector space M_{m x n}(F) = {m x n matrices with entries from F}
Def: If A is in M_{m x n}(F), then A^t = "A transpose" is in F_{n x m}(F)
with (A^t)_ij = A_ji.
Def: If A in M_{n x n}(F) satifies A = A^t, we say A is symmetric

Ex 6 of subspace: Let W = {A: A in M_{n x n}(F) and A symmetric}.
ASK & WAIT Why is W a subspace? Check conditions (1), (2), (3)

Def: if A = -A^t, we say A is skew-symmetric
Ex 7 of subspace: Let W = {skew symmetric matrices}
ASK & WAIT: Why is W a subspace?
ASK & WAIT: if A in W, what is A_ii? (trick question)

Ex 8 of subspace: W = {A in M_{m x n}(F) where A_ij = 0_F unless i=j}
                    = {diagonal matrices}
ASK & WAIT Why is W a subspace?

Def: If A in M_{n x n}(F) then tr(A) = "trace of A" = sum_{i=1 to n} A_ii

Ex 9 of subspace: W = {A in M_{n x n}(F) where tr(A) = 0_F}
ASK & WAIT: Why is W a subspace?

Later we will show that tr(A) = sum of all the eigenvalues of A
This will let us conclude that
   if sum of A's eigenvalues = 0_F, and sum of B's eigenvalues = 0_F,
   then the sum of (A+B)'s eigenvalues = 0_F

Def: Let V be a vector space over field F, v_1,...,v_n vectors in V
     and f_1,...,f_n in F. Then the finite sum
     v = sum_{i=1 to n} f_i*v_i  is called a linear combination of v_1,...,v_n
     (note: ok to have some v_i repeated, eg 1*x + (-1)*x )

Def: Let S = subset of vector space V
     then span(S) = set of all linear combinations of vectors in S

Ex 1 of span(S): V = F^3, S = {(1,0,0), (0,1,0)}
        span(S) = {f_1*(1,0,0) + f_2*(0,1,0), f_1,f_2 in F}
                = {(f_1,f_2,0), f_1,f_2 in F}
ASK & WAIT: What is span(S), geometrically?
            What other property does span(S) have?
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Thm 3: Given vector space V over field F, and subset S of V, then
       span(S) is a subspace of V. If W is any subspace containing S,
       then W also contain span(S) (In other words, span(S) is the
       smallest subspace containing S)
Proof: confirm 3 conditions to show span(S) is a subspace
       (1) 0_V = 0_F * x for any x in S by Thm last time,
           so 0_V is in span(S)
       (2) if sum_i f_i*s_i and sum_j g_j*t_j are in span(S)
           (with f_i and g_j in F and s_i and t_j in S), then
           their sum is clearly of the same form and so in span(S)
       (3) if sum_i f_i*s_i is in span(S) and t in F, then
           t * sum_i f_i*s_i = sum_i (t*f_i)*s_i is in span(S) too
           (use distributivity, associativity, many times!)
       If W contains S, it contains all linear combinations of vectors in S
       (since W is closed under + and *), namely span(S)

Ex: This shows that span(S) in last example really is a subspace

Def: if W = span(S), we say S generates (or spans) W

A common question is this: given S and another vector v, is v in span(S)?
You actually know how to answer this question from Math 54 because we
can reduce it to solving a system of linear equations.

Ex: V = P(Q) = polynomials with rational coefficients
    S = {s1,s2} = {x^3 + 2x^2 +2x+3, x^3+x^2+3x+4}
    is v = x^3 + 4x^2 +1 in span(S), i.e. are there numbers a,b such that
    v = a*s1+b*s2 or
    x^3 + 4x^2 +1 = a*(x^3 + 2x^2 +2x+3) + b*(x^3+x^2+3x+4)
                  = (1*a + 1*b)*x^3 + (2*a+1*b)*x^2 + (2*a+3*b)*x + (3*a+4*b)
    or  1*a + 1*b = 1
        2*a + 1*b = 4
        2*a + 3*b = 0
        3*a + 4*b = 1
ASK & WAIT: Have you seen such systems from Math 54? How do you solve them?
    By applying techniques from Math 54, this turns into
        1*a +    0*b = 3
        0*a + (-1)*b = 2
        0*a +    0*b = 0
        0*a +    0*b = 0
    Eq 1 says a=3, Eq 2 says b = -2, last 2 equations true for all a,b
    So there is a solution, and v is in span(S)
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    We can also write this using matrix notation (which we will review)
      [ 1  1 ] * [ a ] = [ 1 ]   =>   [ 1  0 ] * [ a ] = [ 3 ]
      [ 2  1 ]   [ b ]   [ 4 ]        [ 0 -1 ]   [ b ]   [ 2 ]
      [ 2  3 ]           [ 0 ]        [ 0  0 ]           [ 0 ]
      [ 3  4 ]           [ 1 ]        [ 0  0 ]           [ 0 ]
ASK & WAIT: what if v = x^3 + 4x^2 + 2? is v in span(S)?

Ex: P(F) as above, S = {x+2, x+3, 2x+2}, v = 5, is v in span(S)?
    Seek a, b, c such that a*(x+2) + b*(x+3) + c*(2x+2) = 5 or
      1*a + 1*b + 2*c = 0
      2*a + 3*b + 2*c = 5
ASK & WAIT: How does this differ from last set of equations?
    By applying techniques from Math 54, this turns into
      1*a + 0*b + 4*c = -5
      0*a + 1*b - 2*c =  5
    or
      a = -5 + (-4)*c
      b =  5 +    2*c
ASK & WAIT: What is the solution? What is the solution geometrically?

Ex: P(F) as above, S as above, v = x^2, is v in span(S)?
ASK & WAIT: Yes or no? How many equations in how many unknowns do you get?

So we have three different situations in finding if v in span(S)
  (a) v is not in span(S)
  (b) v is unique linear combination of vectors in S
  (c) v can be as written infinitely many different linear combinations
    (assuming F has characteristic 0!)
ASK & WAIT: why are these all the possibilities?

Here is a theorem that says when you must be in the simpler cases (a) or (b)

Def: The subset S in V is called linearly independent if
for any subset s1,...,sn of S, the only solution to
(*)  0_V = f1*s1 + f2*s2 + ... + fn*sn
is f1=f2=...=fn=0_F. We also say the vectors in S are linearly independent,
or just independent.
Otherwise, if 0_V can be written as (*) with some si in S and
some nonzero fi, we say S is linearly dependent.

Thm: Let V be a vector space over F, S a subset.
Then any v in span(S) can be written
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as a unique linear combination of vectors in S if and only if
S is linearly independent.

Proof:
First assume there were some v expressible in 2 different ways;
we will show S is not independent (dependent).
   v = sum_i f_i*s_i and v = sum_i g_i*s_i
By adding terms of the form 0_F*s_i to either sum, we can ensure that
both sums are over the same subset s_1,...,s_n of S. Subtract these two
sums to get
   0_V = sum_i f_i*s_i - sum_i g_i*s_i = sum_i (f_i-g_i)*s_i
Since the sums were different, not all f_i-g_i=0_F,
so S is dependent.

Now assume S is dependent, so that
   0_V = sum_i f_i*s_i
for some nonzero f_i. Then any nonzero v in span(S) can be written
in two ways, as v and as v + sum_i f_i*s_i

Ex: S = {s1,s2} = {x^3 + 2x^2 +2x+3, x^3+x^2+3x+4} from before.
ASK & WAIT: Is S independent or dependent?

Ex: S = {x+2, x+3, 2x+2}
ASK & WAIT: Is S independent or dependent?

Def: If S is linearly independent then we say that S is a basis
     for V = span(S)