Goal for the day: Recall def of vector space
Define subspace, give examples, basic properties

Def Let V be a vector space over field F. A subset W of V is called a subspace of V if it is also a vector space, with the same operations + and * used for V.

EX; V is a subspace of V, \{0_V\} is a subspace of V, called the zero subspace

To prove that subset W of V is a subspace, we need to confirm that axioms VS1 through VS8 all hold. Fortunately, VS1, VS2, VS5, VS6, VS7 and VS8 (commutativity, associativity, distributivity, mult by 1) all hold automatically, because they hold for V.

So we only need to worry about
1) Is W closed under addition?: for all x,y in W, x+y is in W
2) Is W closed under scalar mult?: for all x in W and t in F, t*x is in W
3) W contains a zero vector, called 0_W (VS3)
4) if x is in W so is -x (VS4)

In fact, it is enough to check (1), (2) and (3) because (4) can be shown to be true if (1) and (2) are true:

Thm 1: Let W be a subset of vector space V (over field F).
Then W is a subspace of V if and only if
1) 0_V is in W
2) for all x and y in W, x+y is in W
3) for all x in W and t in F, t*x is in W

Proof: We first assume W is a subspace, and then prove (1), (2) and (3) hold.
(2) and (3) hold by the definition of subspace.
To prove (1), let 0_W be the zero vector for W; we have to show 0_W = 0_V.
But for any w in W, w + 0_W = w = w + 0_V, by the defs of 0_W and 0_V.
Applying the Cancellation Law, 0_W = 0_V

Next we assume (1), (2) and (3) hold, and show W is a subspace.
(2) and (3) imply that W is closed under + and * as required.
VS3 (existence of 0_W in W) holds for W because we can set 0_W = 0_V.
VS4 (-x is in W for all x in W) holds for W because a Theorem from last time (Thm 1.2, section 1.2) says -x = (-1)*x, which is in W by (3)
It is reasonable to ask whether we really need part (1): Why not just use part (3) with \( t = 0_F \) to conclude that \( 0_F \cdot x = 0_V \) is in \( W \)? Here is why. \( W = \text{null set} \) satisfies (2) and (3), but not (1) ((2) and (3) are "vacuously" true). But the null set is not a vector space, because one of the requirements of a vector space is that it contain a zero vector.

Ex 1 of subspace: Let \( V = F^2 = \{ (x, y), \ x \text{ and } y \in F \} \), and let \( W = \{ (x, 0_F), \ x \in F \} = \text{x axis} \)
ASK & WAIT: why is \( W \) a subspace?

Ex 2 of subspace: Let \( V = F^2 \), choose any \( f \) in \( F \), let \( W_f = \{ (f \cdot x, x), \ x \in F \} \)
ASK & WAIT: why is \( W_f \) a subspace? What does it look like, geometrically, as a subset of \( F^2 \)?

Ex 3 of subspace: Let \( V = F^3 \), choose any nonzero \( f = (f_1, f_2, f_3) \) in \( V \), let \( W_f = \{ (f_1 \cdot x, f_2 \cdot x, f_3 \cdot x), \ x \in F \} \)
ASK & WAIT: why is \( W_f \) a subspace? What does it look like, geometrically?

Ex 4 of subspace: Let \( V = F^3 \), choose nonzero \( f = (f_1, f_2, f_3) \) in \( V \), let \( W'_f = \{ (x_1, x_2, x_3) \text{ such that } x_1 \cdot f_1 + x_2 \cdot f_2 + x_3 \cdot f_3 = 0_F \} \)
ASK & WAIT: Why is \( W'_f \) a subspace? What does it look like geometrically?

Ex 5 of subspace: Let \( V = F^3 \), choose nonzero \( f = (f_1, f_2, f_3) \) and \( g = (g_1, g_2, g_3) \) in \( V \), let \( W'_{f,g} = \{ (x_1, x_2, x_3) \text{ such that } x_1 \cdot f_1 + x_2 \cdot f_2 + x_3 \cdot f_3 = 0_F = x_1 \cdot g_1 + x_2 \cdot g_2 + x_3 \cdot g_3 \} \)
ASK & WAIT: Why is \( W'_{f,g} \) a subspace? What does it look like geometrically?

Note: We can also write \( W'_{f,g} = W'_{f} \cap W'_{g} \)

This is a special case of

Thm 2: An intersection of subspaces of \( V \) is also a subspace
Proof: Let \( C \) be a set of subspaces of \( V \). Let \( W \) be the intersection of all these subspaces. We need to confirm that the 3 conditions of Thm 1 apply to \( W \).
(1) let \( W_c \) be any of the subspaces in \( C \). Then since \( 0_V \) is in \( W_c \), it is in their intersection, namely \( W \).
(2) Let \( x \) and \( y \) be in \( W \). Then they must also be in \( W_c \), so \( x+y \) is in \( W_c \). Since this is true for all \( W_c \), \( x+y \) is in their intersection, namely \( W \)
(3) same idea as (2)
Recall vector space $M_{m \times n}(F) = \{m \times n \text{ matrices with entries from } F\}$

Def: If $A$ is in $M_{m \times n}(F)$, then $A^t = "A \text{ transpose}"$ is in $F_{n \times m}(F)$ with $(A^t)_{ij} = A_{ji}$.

Def: If $A$ in $M_{n \times n}(F)$ satifies $A = A^t$, we say $A$ is symmetric

Ex 6 of subspace: Let $W = \{A: A \text{ in } M_{n \times n}(F) \text{ and } A \text{ symmetric}\}$.
ASK & WAIT Why is $W$ a subspace? Check conditions (1), (2), (3)

Def: if $A = -A^t$, we say $A$ is skew-symmetric

Ex 7 of subspace: Let $W = \{\text{skew symmetric matrices}\}$
ASK & WAIT: Why is $W$ a subspace?
ASK & WAIT: if $A$ in $W$, what is $A_{ii}$? (trick question)

Ex 8 of subspace: $W = \{A \text{ in } M_{m \times n}(F) \text{ where } A_{ij} = 0_F \text{ unless } i=j\}$

= \{\text{diagonal matrices}\}

ASK & WAIT Why is $W$ a subspace?

Def: If $A$ in $M_{n \times n}(F)$ then $\text{tr}(A) = \"trace of } A = \sum_{i=1 \text{ to } n} A_{ii}

Ex 9 of subspace: $W = \{A \text{ in } M_{n \times n}(F) \text{ where } \text{tr}(A) = 0_F\}$
ASK & WAIT: Why is $W$ a subspace?

Later we will show that $\text{tr}(A) = \sum$ of all the eigenvalues of $A$

This will let us conclude that

if sum of $A$’s eigenvalues $= 0_F$, and sum of $B$’s eigenvalues $= 0_F$, then the sum of $(A+B)$’s eigenvalues $= 0_F$

Def: Let $V$ be a vector space over field $F$, $v_1,\ldots,v_n$ vectors in $V$
and $f_1,\ldots,f_n$ in $F$. Then the finite sum
$v = \sum_{i=1 \text{ to } n} f_i*v_i$ is called a linear combination of $v_1,\ldots,v_n$

(note: ok to have some $v_i$ repeated, eg $1*x + (-1)*x$)

Def: Let $S = \text{subset of vector space } V$
then $\text{span}(S) = \text{set of all linear combinations of vectors in } S$

Ex 1 of $\text{span}(S)$: $V = F^3$, $S = \{(1,0,0), (0,1,0)\}$

$\text{span}(S) = \{f_1*(1,0,0) + f_2*(0,1,0), f_1,f_2 \text{ in } F\}$

= $\{(f_1,f_2,0), f_1,f_2 \text{ in } F\}$

ASK & WAIT: What is $\text{span}(S)$, geometrically?

What other property does $\text{span}(S)$ have?
Thm 3: Given vector space $V$ over field $F$, and subset $S$ of $V$, then \( \text{span}(S) \) is a subspace of $V$. If $W$ is any subspace containing $S$, then $W$ also contain $\text{span}(S)$ (In other words, $\text{span}(S)$ is the smallest subspace containing $S$)

Proof: confirm 3 conditions to show $\text{span}(S)$ is a subspace

1. $0_V = 0_F \times x$ for any $x$ in $S$ by Thm last time, so $0_V$ is in $\text{span}(S)$

2. if $\sum_i f_i s_i$ and $\sum_j g_j t_j$ are in $\text{span}(S)$ (with $f_i$ and $g_j$ in $F$ and $s_i$ and $t_j$ in $S$), then their sum is clearly of the same form and so in $\text{span}(S)$

3. if $\sum_i f_i s_i$ is in $\text{span}(S)$ and $t$ in $F$, then $t \times \sum_i f_i s_i = \sum_i (t f_i) s_i$ is in $\text{span}(S)$ too (use distributivity, associativity, many times!)

If $W$ contains $S$, it contains all linear combinations of vectors in $S$ (since $W$ is closed under $+$ and $\times$), namely $\text{span}(S)$

Ex: This shows that $\text{span}(S)$ in last example really is a subspace

Def: if $W = \text{span}(S)$, we say $S$ generates (or spans) $W$

A common question is this: given $S$ and another vector $v$, is $v$ in $\text{span}(S)$? You actually know how to answer this question from Math 54 because we can reduce it to solving a system of linear equations.

Ex: $V = P(Q) = \text{polynomials with rational coefficients}$

$S = \{s_1, s_2\} = \{x^3 + 2x^2 + 2x + 3, x^3 + x^2 + 3x + 4\}$

is $v = x^3 + 4x^2 + 1$ in $\text{span}(S)$, i.e. are there numbers $a,b$ such that $v = a s_1 + b s_2$ or

$x^3 + 4x^2 + 1 = a(x^3 + 2x^2 + 2x + 3) + b(x^3 + x^2 + 3x + 4)$

or $1a + 1b = 1$

$2a + 1b = 4$

$2a + 3b = 0$

$3a + 4b = 1$

ASK & WAIT: Have you seen such systems from Math 54? How do you solve them?

By applying techniques from Math 54, this turns into

$1a + 0b = 3$

$0a + (-1)b = 2$

$0a + 0b = 0$

$0a + 0b = 0$

Eq 1 says $a=3$, Eq 2 says $b = -2$, last 2 equations true for all $a,b$

So there is a solution, and $v$ is in $\text{span}(S)$


We can also write this using matrix notation (which we will review)
$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

ASK & WAIT: what if \( v = x^3 + 4x^2 + 2 \) is \( v \) in \( \text{span}(S) \)?

Ex: \( P(F) \) as above, \( S = \{x+2, x+3, 2x+2\} \), \( v = 5 \), is \( v \) in \( \text{span}(S) \)?
Seek \( a, b, c \) such that \( a*(x+2) + b*(x+3) + c*(2x+2) = 5 \) or

\[
\begin{align*}
1*a & + 1*b + 2*c = 0 \\
2*a & + 3*b + 2*c = 5
\end{align*}
\]

ASK & WAIT: How does this differ from last set of equations?

By applying techniques from Math 54, this turns into

\[
\begin{align*}
1*a & + 0*b + 4*c = -5 \\
0*a & + 1*b - 2*c = 5
\end{align*}
\]

or

\[
\begin{align*}
a & = -5 + (-4)*c \\
b & = 5 + 2*c
\end{align*}
\]

ASK & WAIT: What is the solution? What is the solution geometrically?

Ex: \( P(F) \) as above, \( S \) as above, \( v = x^2 \), is \( v \) in \( \text{span}(S) \)?
ASK & WAIT: Yes or no? How many equations in how many unknowns do you get?

So we have three different situations in finding if \( v \) in \( \text{span}(S) \)
(a) \( v \) is not in \( \text{span}(S) \)
(b) \( v \) is unique linear combination of vectors in \( S \)
(c) \( v \) can be as written infinitely many different linear combinations
(assuming \( F \) has characteristic 0!)

ASK & WAIT: Why are these all the possibilities?

Here is a theorem that says when you must be in the simpler cases (a) or (b)

Def: The subset \( S \) in \( V \) is called linearly independent if

for any subset \( s_1, \ldots, s_n \) of \( S \), the only solution to

\[ (*) \quad 0_V = f_1*s_1 + f_2*s_2 + \ldots + f_n*s_n \]

is \( f_1 = f_2 = \ldots = f_n = 0_F \). We also say the vectors in \( S \) are linearly independent, or just independent.

Otherwise, if \( 0_V \) can be written as \((*)\) with some \( s_i \) in \( S \) and some nonzero \( f_i \), we say \( S \) is linearly dependent.

Thm: Let \( V \) be a vector space over \( F \), \( S \) a subset.
Then any \( v \) in \( \text{span}(S) \) can be written
as a unique linear combination of vectors in $S$ if and only if $S$ is linearly independent.

Proof:
First assume there were some $v$ expressible in 2 different ways; we will show $S$ is not independent (dependent).

$v = \sum_i f_i s_i$ and $v = \sum_i g_i s_i$

By adding terms of the form $0_F s_i$ to either sum, we can ensure that both sums are over the same subset $s_1, \ldots, s_n$ of $S$. Subtract these two sums to get

$0_V = \sum_i f_i s_i - \sum_i g_i s_i = \sum_i (f_i - g_i) s_i$

Since the sums were different, not all $f_i - g_i = 0_F$, so $S$ is dependent.

Now assume $S$ is dependent, so that

$0_V = \sum_i f_i s_i$

for some nonzero $f_i$. Then any nonzero $v$ in span($S$) can be written in two ways, as $v$ and as $v + \sum_i f_i s_i$

Ex: $S = \{s_1, s_2\} = \{x^3 + 2x^2 + 2x + 3, x^3 + x^2 + 3x + 4\}$ from before.
ASK & WAIT: Is $S$ independent or dependent?

Ex: $S = \{x+2, x+3, 2x+2\}$
ASK & WAIT: Is $S$ independent or dependent?

Def: If $S$ is linearly independent then we say that $S$ is a basis for $V = \text{span}(S)$