Math 110 - Fall 05 - Lectures notes # 1 - Aug 29 (Monday)

Name, class, URL (www.cs.berkeley.edu/~demmel/ma110) on board
See Barbara Peavy in 967 Evans Hall for all enrollment issues.
All course material will be on the web page. If this is
inadequate, please send me email and I will put copies at
Copy Central Northside.
Read Course Outline on web page for course rules and grading policies
You are responsible for reading this and knowing the rules!

Text, which we will follow fairly closely:
Linear Algebra, 4th Edition, by Friedberg/Insel/Spence,
same as last semester.

Topics (following table of contents):
- Chap 1: Vector spaces ("vectors" and their basic properties)
- Chap 2: Linear transformations ("matrices" and their basic properties)
- Chap 3: Elementary Matrix Operations (quickly)
- Chap 4: Determinants
- Chap 5: Diagonalization (eigenvalues and vectors)
- Chap 6: Inner products, orthgonality, singular value decomposition
- Chap 7: Jordan Form (generalizations of eigenvectors)

Examples (why is linear algebra important)

Linear Equations: Consider new Bay Bridge: how is it designed?
How do they know it will be strong enough to hold up the traffic?
If you imagine a car sitting on the end of a beam, you
can write a simple relationship

F (force from weight of car on bridge)
= k (stiffness of beam) * x (how much beam bends)

So if you know the weight of the car, and the stiffness of the beam,
you can solve one linear equation in one unknown F=kx for
x = F/k to get the amount the bridge bends, and compare that
to how much you are willing to let it bend. In a real bridge
there are many cars, many beams, and many x's describing how each
beam bends, and the the beams are connected. So instead of
of 1 linear equation in 1 unknown, you get thousands of linear
equations in thousands of unknowns, which civil engineers have to solve.
The same process applies to any important structures
(buildings, airplanes, cars, certain computer chips ...).
See CE 130, ME 104.

All radio, light and other electromagnetic radiation in free space satisfy a system of linear equations called Maxwell's equations. These are partial differential equations, but linear nonetheless. See Ph 7B or Ph 110.

EG: Aim 2 flashlights at board and observe that lit spot 1 is same whether or not beam 2 passes through beam 1.

ASK & WAIT: Can you explain this using linear algebra?

Eigenvalues and Eigenvectors:

ASK & WAIT: Each of you depends on an eigenvector of one of the world's largest matrices, many times each day. How big is the matrix? Hint: what number do you see when you go to www.google.com?

Schroedinger equation: every atom, molecule, your physical body is most acccurately described as an "eigenvector" of Schroedinger's equation, which is a partial differential equation. The eigenvalues correspond to energy levels, and the eigenvectors describe how the electrons are distributed around all the atomic nuclei. See Ph 7C or Ph 137.

What we will cover: We will concentrate on definitions, and theorems describing basic properties of these linear algebra objects like linear transformations, their inverses (when they exist), eigenvalues and eigenvectors (when they exist).
In particular, you will practice reading and writing clear and correct mathematical proofs.

We will also try to look at concepts and problem solving from at least 2 points of view: algebra and geometry, because linear algebra natually incorporates and can be understood both ways:
EG: One can either say:

2 equations in 2 unknowns can	or	2 lines in the plane can
(1) have a unique solution		(1) intersect in a point
(2) have no solutions		(2) be parallel and not intersect
(3) have infinitely many solutions		(3) be identical

What we will not cover: practical algorithms for solving linear equations, finding eigenvalues/vectors (Ma128ab, Ma221)
This course is a prerequisite for such courses.

Prereq: Math 54, including definitions of sets, functions (Apps A, B)

Homework will be due Thursdays at the start of section. There will be brief weekly quizzes to make sure people are keeping up.

To get started, let's talk carefully about the numbers that we will be writing in our vectors and matrices. As motivation:

Consider solving 2*x = 1: even though there are only integers in the equation, we can't solve for x if we only look for x in Z = set of integers, we need Q = set of rational numbers.

Consider solving x^2-2 = 0; even though there are only rational numbers in the equation, we can't solve for rational x, we need real numbers R

Consider solving x^2 + pi = 0; even though there are only real numbers in the equation, we can't solve for real x, we need complex numbers C.

DEF: A set of numbers (also called scalars) F is called a field if there are two binary operations + (addition) and * (multiplication) so that x+y and x*y are unique numbers in F for all x and y in F, and such that + and * satisfy the following conditions:

(1) for all x and y in F: x+y=y+x and x*y=y*x
(commutativity of addition and multiplication)

(2) for all x, y, z in F: (x+y)+z = x+(y+z) and (x*y)*z = x*(y*z)
(associativity of addition and multiplication)

(3) There exist distinct scalars 0 and 1 such that
for all x in F: x+0=x and x*1=x
(existence of identity elements for addition, multiplication)

(4) For all x in F and nonzero y in F, there exist x' and y' such that
x + x' = 0 and y*y' = 1. We denote x' by -x, and y' by y^(-1).
(existence of inverses for addition, multiplication)

(5) for all x,y,z in F: x*(y+z) = x*y + x*z
(distributivity)

ASK & WAIT: Is Z a field? Why?

ASK & WAIT: Is Q a field?

ASK & WAIT: Is R a field?

ASK & WAIT: Is C a field?

ASK & WAIT: Is S = {q1 + q2*sqrt(2), q1 and q2 in Q} a field?

ASK & WAIT: Is A = {all roots of polynomials with integer coeffs} a field?

EG: Z2 = {0,1} with 0+0=0, 0+1=1, 1+1=0; 0*0=0, 0*1=0, 1*1=1
Can show this is a field (homework!)
ASK & WAIT: Does this field, with its operations, have other names?

EG: Zp = {0,1,2,...,p-1} where p is a prime,
       x+y = x+y mod p = remainder after dividing x+y by p
       x*y = x*y mod p = remainder after dividing x*y by p
Can show this is a field (homework!)
Used in cryptography (see Ma55)

DEF: A field F has characteristic p if 1+1+...+1 (p times) = 0 for some
positive integer p. Otherwise F has characteristic 0.

ASK & WAIT: What is the characteristic of Q?
ASK & WAIT: What is the characteristic of R, C, S, A?
ASK & WAIT: What is the characteristic of Z2?
ASK & WAIT: What is the characteristic of Zp?

EG: F(x) = rational functions in x with coefficients from another field F
of characteristic 0. So fields don't have to be "numbers" in the usual sense!

Some of the results in the book are for any field, some are only for
characteristic 0. We will try to carefully distinguish, but when in
doubt (eg for homework) assume characteristic 0 unless told otherwise.

The point of fields is that all the familiar rules of algebra just work.
In other words, given the 5 properties above, other familiar ones follow:

Thm (Cancellation Laws)
  (a) for all x,y,z in field F, x+z = y+z implies x=y
  (b) for all x,y, nonzero z in field F, x*z = y*z implies x=y

Proof: (a) homework!
  (b) z nonzero => exists z' such that z*z' = 1. then

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      x*z = y*z => (x*z)*z' = (y*z)*z'   by def of mult
         => x*(z*z') = y*(z*z')          by associativity of mult
         => x*1 = y*1                    by def of z'
         => x = y                        by def of 1
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Thm: (uniqueness of 0,1, inverses)
     (a) 0 and 1 from part (3) are unique,

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     (b) x' and y' from part (4) are unique

Proof:
     (a): suppose there are two zeros, 0 and 0'. To show they must
          be equal, note that 0+0 = 0 by def of 0 and 0 + 0' = 0 by def of 0'
          Apply cancellation law to 0+0 = 0+0' = 0'+0.
ASK & WAIT: how do we prove (b)?

Thm: a*0 = 0
Proof: 0 + a*0 = a*0         by def of 0
               = a*(0 + 0)   by def of 0
               = a*0 + a*0   by distributivity
       so 0 = a*0 by part (a) of Cancellation Law

Def Subtraction a-b is defined as a + (-b), where -b is additive inverse of b

Def Division a/b for nonzero b is defined as a * (b^(-1)),
    where b^(-1) is multiplication inverse of b

Thm 0^(-1) does not exist (i.e. can't divide by zero)
Proof: by last thm, there is no a such that a*0 = 1


   First homework: Review Apps A, B; Read App C, Read Chap 1.1-1.4
      Due: Thursday Sep 8:
        (1) Prove part (a) of Cancellation Law:
            for all x,y,z in field F, x+z = y+z implies x=y
        (2) Prove Z2 is a field. Hint: you can either make tables
             of all possible values needed to confirm the properties in
             the definition, or claim it as a special case of the
             next problem
        (3) (Extra credit) Prove Zp is a field. Hint: To show there is
            a multiplicative inverse of any nonzero b, apply
            pigeon-hole principle to set of p-1 numbers
               1*b mod p , 2*b mod p,..., (p-1)*b mod p
        (4) Sec 1.2: 1 (justify your answers),
                     7, 9,
                     12 (but for odd functions, f(-t) = -f(t), not even)
                     15, 16, 19,
        (5) Sec 1.3: 1 (justify your answers)
                     2d, 5, 8, 9,
                     11 (as stated, and also changed to read
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                  "f(x)=0 and f(x) has degree <= n")
             12, 15, 23, 28
(6) Sec 1.4: 1 (justify your answers)
             13
(7) Sec 1.5: 1 (justify your answers)
             12
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