MATH 110: LINEAR ALGEBRA
HOMEWORK #10

Problem (1). *Use induction on the dimension n of A to prove that* $\det(A - xI)$ *is a polynomial in x of degree n, with highest degree term $(-1)^n x^n$.*

We want to use induction applied to the determinant expansion of $A - xI$ along its first row. But we have to choose the induction hypothesis carefully, because dropping the first row and column j of $A - xI$ to get $\widetilde{(A - xI)}_{1j}$ does *not* yield a matrix with $-x$s subtracted from its diagonal. Instead, the induction hypothesis is that if each entry of an n by n matrix $B + xC$ is a polynomial in x of degree 0 or 1, then the determinant of $B + xC$ is a polynomial in x of degree at most n.

If $n = 1$, then $\det(B - xC) = B_{11} - xC_{11}$, which is indeed a polynomial in x of degree 0 (if $C_{11} = 0$) or 1 (otherwise). Assuming that the statement holds for n, we'll prove it for $n + 1$. We compute $\det(B - xC)$ by expanding along the first row:

$$\det(B - xC) = \sum_{j=1}^{n+1} (-1)^{1+j} (B - xC)_{1j} \det(\widetilde{(B - xC)}_{1j}) = \sum_{j=1}^{n+1} (-1)^{1+j} (B_{1j} - xC_{1j}) \det(\tilde{B}_{1j} - x\tilde{C}_{1j})$$

By induction the degree of each $\det(\tilde{B}_{1j} - x\tilde{C}_{1j})$ is at most n, so the degree of $(B_{1j} - xC_{1j})\det(\tilde{B}_{1j} - x\tilde{C}_{1j})$ is at most $n + 1$, so the degree of $\det(B - xC)$ is at most $n + 1$, as desired.

To see that the highest order term in x is $(-1)^n x$, we again use induction, but applied to $B - xC = A - xI$:

$$\begin{aligned} \det(A - xI) &= \sum_{j=1}^{n+1} (-1)^{1+j} (A_{1j} - xI_{1j}) \det(\tilde{A}_{1j} - x\tilde{I}_{1j}) \\ &= (A_{11} - x)\det(\tilde{A}_{11} - x\tilde{I}_{11}) + \sum_{j=2}^{n+1} (-1)^{1+j} A_{1j} \det(\tilde{B}_{1j} - x\tilde{C}_{1j}) \end{aligned}$$

The sum from $j = 2$ to $n + 1$ has degree at most n by the previous result. By induction $\det(\tilde{A}_{11} - x\tilde{I}_{11}) = (-1)^n x^n +$ lower order terms, so

$$(A_{11} - x)\det(\tilde{A}_{11} - x\tilde{I}_{11}) = (A_{11} - x)((-1)^n x^n + \text{ lower order terms}) = (-1^{n+1} x^{n+1} + \text{ lower order terms}$$

as desired.

§5.1: Eigenvalues and Eigenvectors

Problem 1.

(a) False. For instance, the only eigenvalue of I_n is 1.
(b) True. Any scalar multiple of an eigenvector is also an eigenvector.
(c) True. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{R})$. Then $\det(A - tI) = t^2 + 1$, which has no roots over the real numbers. So A has no eigenvalues, and hence no eigenvectors.

(d) False. Zero is an eigenvalue of A whenever $\det(A) = 0$, by Theorem 5.2.
(e) False. See part (b) above.
(f) False. In Example 4 on p. 248, 3 and -1 are eigenvalues, but 2 is not.
(g) False. The identity linear operator always has 1 as an eigenvalue.
(h) True. This follows from Theorem 5.1.
(i) True. Suppose that $A = Q^{-1}BQ$ are square matrices and $Av = \lambda v$ for some scalar λ and vector v. Then $B(Qv) = QAv = \lambda(Qv)$. So if λ is an eigenvalue of A, then it's an eigenvalue of B.
(j) False. For instance, let A be as in Example 6 on p. 250, and let $B = Q^{-1}AQ$, with Q as on p. 251. Then $(1, 2)$ is an eigenvector of A, as computed in the example. But, $(1, 2)$ is not an eigenvector of B, since the only eigenvalues of B are 3 and -1, and neither is an eigenvalue corresponding to $(1, 2)$.
(k) False. In Example 6 on p. 250, $(1, 2)$ and $(1, -2)$ are eigenvectors, but $(1, 2) + (1, -2) = (2, 0)$ is not.

Problem 3(b). We have $\det(A - tI) = -t^3 + 6t^2 - 11t + 6$. We notice that this polynomial is zero when $t = 1$, so, using polynomial division, we conclude that $\det(A - tI) = (t - 1)(-t^2 + 5t - 6) = -(t - 1)(t - 2)(t - 3)$. Hence, the eigenvalues are 1, 2, and 3. Let $t = 1$, then $A - 1I = \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix}$. Now $\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ leads to a system of three equations in three unknowns. We obtain $\left\{ a \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$ as the set of solutions to this system, which is the set of eigenvectors corresponding to the eigenvalue 1. Performing similar computations for $t = 2$ and $t = 3$, we conclude that the corresponding sets of eigenvectors are $\left\{ a \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}$ and $\left\{ a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} : a \in \mathbb{R} \right\}$, respectively. The set $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is clearly linearly independent, and hence it is a basis for R^3. Setting $Q = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, we have $Q^{-1}AQ = D$.

Problem 14. First, note that $(A - tI_n)^t = A^t - (tI_n)^t = A^t - tI_n$, by Exercise 1.3.3. Hence, $\det(A^t - tI_n) = \det((A - tI_n)^t) = \det(A - tI_n)$, by Theorem 4.8. So A and A^t have the same characteristic polynomial, by the definition of characteristic polynomial.

Problem 15.

(a) We proceed by induction on m. Since $Tx = \lambda x$, the statement is true for $m = 1$. In general, $T^m x = T(T^{m-1}x) = T(\lambda^{m-1}x) = \lambda^{m-1}Tx = \lambda^{m-1}\lambda x = \lambda^m x$.
(b) Exactly the same calculation as above, with a matrix A in place of T, gives the desired result.

Problem 17.

(a) Suppose that λ is an eigenvalue of T. Then $T(A) = \lambda A$ for some nonzero matrix A. Hence, $A^t = \lambda A$, that is $A_{ij} = \lambda A_{ji}$ for all i, j with $1 \le i, j \le n$. Since A is nonzero, there are i and j such that $A_{ij} \neq 0$. Then $A_{ij} = \lambda A_{ji} = \lambda(\lambda A_{ij}) = \lambda^2 A_{ij}$, and hence, $\lambda = \pm 1$.

(b) The eigenvectors corresponding to $\lambda = 1$ are the symmetric matrices, while the eigenvectors corresponding to $\lambda = -1$ are the skew-symmetric matrices (i.e., matrices satisfying $A^t = -A$).

(c) By Theorem 5.1 and (b), we just need to find a basis β consisting of symmetric and antisymmetric matrices. Here is one possibility:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

(d) Using the same idea as in (c), we can take β to be the basis consisting of the n matrices E_{ii}, the $(n^2 - n)/2$ matrices $E_{ij} + E_{ji}$ $(1 \le i < j \le n)$, and the $(n^2 - n)/2$ matrices $E_{ij} - E_{ji}$ $(1 \le i < j \le n)$.

Problem 18.

(a) Over the field $\mathbb{C}$, every nonconstant polynomial has a root. In particular, $\det(AB^{-1} - tI)$ has a root, say $t = -c$. So $\det(AB^{-1} + cI) = 0$, and hence $AB^{-1} + cI$ is not invertible. It follows that $A + cB$ is not invertible, since $\det(A + cB) = \det((AB^{-1} + cI)B) = \det(AB^{-1} + cI)\det(B) = 0$.

(b) Here is one possibility: $A = I_2$ and $B = E_{12}$. The inverse of $I_2 + cE_{12}$ is $I_2 - cE_{12}$

Problem 20. By definition of characteristic polynomial, $f(t) = \det(A - tI_n)$. So $a_0 = f(0) = \det(A)$. Finally, recall that a matrix is invertible if and only if its determinant is not zero. (See the Corollary on p. 223.)

§5.2: Diagonalizability

Problem 1.

(a) False. For example, consider the identity linear operator.

(b) False, cf. Example 4 on p. 265.

(c) True. See the definition of E_λ on p. 264.

(d) True. This follows from Theorem 5.5, since if $0 \neq v \in E_{\lambda_1} \cap E_{\lambda_2}$, then, setting $v_1 = v = v_2$, $\{v_1, v_2\}$ is a linearly dependent set.

(e) True, see p. 272.

(f) False. By the test on p. 269, the characteristic polynomial of T must split in order for T to be diagonalizable.

(g) True. By the test on p. 269, the characteristic polynomial of a diagonalizable linear operator splits and hence has at least one root. Therefore, the linear operator has at least one eigenvalue, by Theorem 5.2.

Problem 2(e). $\det(A - tI) = -x^3 + x^2 - x + 1 = -(x - 1)(x^2 + 1)$, which does not split over $\mathbb{R}$. So, by the test on p. 269, A is not diagonalizable.

Problem 12.

(a) Let x be an eigenvector of T corresponding to $\lambda\ (\neq 0)$. Then $x = T^{-1}Tx = T^{-1}(\lambda x) = \lambda T^{-1}x$, so $T^{-1}x = \lambda^{-1}x$, i.e., x is an eigenvector of T^{-1} corresponding to λ^{-1}. Reversing this calculation, we also conclude that if x is an eigenvector of T^{-1} corresponding to λ^{-1}, then x be an eigenvector of T corresponding to λ.

(b) Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T, and denote the multiplicity of λ_i by $\mathrm{m}(\lambda_i)$. If T is diagonalizable, then, by Theorem 5.9, $\mathrm{m}(\lambda_i) = \dim(E_{\lambda_i})$ for all i. Now, for each i, $\dim(E_{\lambda_i}) = \dim(E_{\lambda_i^{-1}})$, by (a). So, by Theorem 5.7, $\mathrm{m}(\lambda_i^{-1}) \geq \dim(E_{\lambda_i^{-1}}) = \mathrm{m}(\lambda_i)$. But, the characteristic polynomials of T and T^{-1} have the same degree, which equals $\sum_{i=1}^{k} \mathrm{m}(\lambda_i) = \sum_{i=1}^{k} \mathrm{m}(\lambda_i^{-1})$. So, in fact, $\mathrm{m}(\lambda_i) = \mathrm{m}(\lambda_i^{-1})$ for each i, and hence $\mathrm{m}(\lambda_i^{-1}) = \dim(E_{\lambda_i^{-1}})$ for each i. Applying Theorem 5.9 again, we conclude that T^{-1} is diagonalizable.

Problem 18(b). Let C and D be any two diagonal $n \times n$ matrices. Then for all $1 \leq i, j \leq n$, $(CD)_{ij} = \sum_{k=1}^{n} C_{ik}D_{kj} = C_{ii}D_{ij}$ and $(DC)_{ij} = \sum_{k=1}^{n} D_{ik}C_{kj} = D_{ii}C_{ij}$, since $C_{ik} = 0$ and $D_{ik} = 0$ whenever $i \neq k$. Hence, if $i \neq j$, then $(CD)_{ij} = 0 = (DC)_{ij}$, and if $i = j$, then $(CD)_{ij} = C_{ii}D_{ii} = D_{ii}C_{ii} = (DC)_{ij}$. We conclude that $CD = DC$

Now, suppose that A and B are simultaneously diagonalizable. Then there is an invertible matrix Q such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are both diagonal. Hence, $Q^{-1}ABQ = Q^{-1}AQQ^{-1}BQ = Q^{-1}BQQ^{-1}AQ = Q^{-1}BAQ$. Multiplying by Q on the left and Q^{-1} on the right, we see that $AB = BA$.