MATH 110: LINEAR ALGEBRA
HOMEWORK #9

CHU-WEE LIM

(1) We have already proven in the lecture that

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A)\det(C)$$

if A, B and C are $n_1 \times n_1$, $n_1 \times n_2$ and $n_2 \times n_2$ matrices respectively. Now we can use this to prove problem 1. Indeed, suppose X, Y and Z are $n_1 \times n_1$, $n_2 \times n_1$ and $n_2 \times n_2$ matrices respectively. Then we have:

$$\det \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} = \det \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}^t = \det \begin{pmatrix} X^t & Y^t \\ 0 & Z^t \end{pmatrix}$$
$$= \det(X^t)\det(Z^t) = \det(X)\det(Z).$$

(2) The proof is by induction on the size of A. When $n = 1$, there is nothing to prove. Now we assume: if B is an $(n-1) \times (n-1)$ upper-triangular matrix, then $\det(B)$ is the product of the diagonal entries of B.

Let A be an $n \times n$ upper-triangular matrix. The last row of A consists of all zeros except the entry A_{nn}. Hence, expanding along the last row gives:

$$\det(A) = (-1)^{n+n} A_{nn} \det(\tilde{A}_{nn}) = A_{nn} \det(\tilde{A}_{nn}),$$

where $\tilde{A}_{nn}$ is the submatrix obtained by deleting the last row and last column of A. Since $\tilde{A}_{nn}$ is an upper-triangular $(n-1) \times (n-1)$ matrix, its determinant is the product of the diagonal entries $A_{11}A_{22} \ldots A_{n-1,n-1}$. Thus,

$$\det(A) = A_{11}A_{22} \ldots A_{nn},$$

as desired.

For the case of lower-triangular matrix A, we note that A^t is upper-triangular, so we can apply the above result:

$$\det(A) = \det(A^t) = A_{11}A_{22} \ldots A_{nn}.$$

§4.2: Determinants of Order n

Problem 1.

(a) **False**, e.g. for the 2×2 identity matrix I, we have $\det(2I) = 4 \neq 2 = \det(I) + \det(I)$.
(b) **True**, see theorem 4.4 (pg 215).
(c) **True**, for we can subtract one row from the other to get a row of zeros.
(d) **True**, see rule (a) on page 217.
(e) **False**, if we multiply the row by 0, then $\det(B) = 0$ regardless of what A is.

Date: November 1.

(f) **False**, suppose $k = 0$, $A = I$. Then adding $0 \cdot R_2$ to R_1 has no effect, and so $\det(B) = \det(A) = 1 \neq 0 \cdot \det(A)$.
(g) **False**. Quite the opposite: A has rank n if and only if A is invertible if and only if $\det(A) \neq 0$.
(h) **True**: we just proved it above.

Problem 26. Using Q25, we get $\det(-A) = (-1)^n \det(A)$. Hence $\det(A) = \det(-A)$ iff $\det(A) = (-1)^n \det(A)$.

Now, if n is even then $(-1)^n = +1$ so equality clearly holds. Also, if $\mathrm{char}(F) = 2$, then $1_F + 1_F = 0_F$ and so $(-1)^n = \pm 1 = 1$ regardless of the parity of n. Hence equality still holds.

Finally, suppose n is odd and $\mathrm{char}(F) \neq 2$. Then we get $\det(A) = -\det(A)$, which gives $2\det(A) = 0$. Since $\mathrm{char}(F) \neq 2$, we get $\det(A) = 0$. Hence $\det(A) = \det(-A)$ if and only if at least one of the following is true:
(i) $\mathrm{char}(F) = 2$;
(ii) n is even;
(iii) $\det(A) = 0$.

Problem 30. If we exchange two rows of A, then we flip the sign of $\det(A)$. Also, B is obtained from A by exchanging the i-th row and the $(n+1-i)$-th row, for $i = 1, 2, \ldots, [\frac{n}{2}]$. Here, $[x]$ is the greatest integer $\leq x$. Hence, we see that $\det(B) = (-1)^{[\frac{n}{2}]} \det(A)$.

As an alternative, you can also write $(-1)^{\frac{n(n-1)}{2}} \det(A)$. This can be seen by performing $(n-1)$ row-exchanges to move the bottom row to the top; folowed by $(n-2)$ row-exchanges to move the bottom row to the second, and so on. This gives us $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$ row-exchanges.

§4.3: Properties of Determinants

Problem 1.

(a) **False**, an elementary matrix of type (b) is not of determinant 1 in general.
(b) **True**, by theorem 4.7, page 223.
(c) **False**. In fact M is invertible if and only if $\det(M) \neq 0$. See corollary on page 223.
(d) **True**, since M has rank n if and only if it is invertible.
(e) **False**. In fact, $\det(A^t) = \det(A)$ by theorem 4.8, page 224.
(f) **True**, using the fact that we can perform cofactor expansion, and that $\det(A^t) = \det(A)$.
(g) **False**. E.g. $0x = 0$ cannot be solved by Cramer's rule.
(h) **False**. E.g. try solving $x_1 = 2, x_1 + 2x_2 = 0$ by this new Cramer's rule. We get $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right)$ and $M_k = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right)$, and so $\det(M_k)/\det(A) = 0 \neq x_2$.

Problem 10. If M is nilpotent, then $M^k = 0$ for some k. So $\det(M)^k = \det(M^k) = 0$, and hence $\det(M) = 0$.

Problem 11. If $M^t = -M$, then taking the determinant gives $\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M)$. If n is odd, then $\det(M) = -\det(M)$ and so $\det(M) = 0$ (recall that we are working over the complex field $\mathbb{C}$, so char $\neq 2$). Hence M is not invertible.

On the other hand $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is an example of a skew-symmetric invertible 2×2 matrix.

Problem 12. We have $1 = \det(I) = \det(QQ^t) = \det(Q)\det(Q^t) = \det(Q)\det(Q)$. Hence $\det(Q)^2 = 1$ and so $\det(Q) = \pm 1$.

Problem 13. (a) Suppose M has the LU-decomposition: $M = P_1 L U P_2$, where P_1 and P_2 are permutation matrices. Also, L is a unit lower-triangular matrix, while U is an upper-triangular matrix. Then

$$\det(M) = \det(P_1)\det(L)\det(U)\det(P_2) = \det(P_1)U_{11}U_{22}\dots U_{nn}\det(P_2).$$

Then taking the conjugate gives $\overline{M} = \overline{P}_1\overline{LU}\overline{P}_2 = P_1\overline{LU}P_2$. Hence

$$\det(M) = \det(P_1)\det(\overline{U})\det(P_2) = \det(P_1)\overline{U}_{11}\overline{U}_{22}\dots\overline{U}_{nn}\det(P_2) = \overline{\det(M)},$$

since $\det(P_1)$ and $\det(P_2)$ are ± 1.

Alternative solution: use induction on the size of M.

(b) We have $1 = \det(I) = \det(QQ^*) = \det(Q)\det(\overline{Q}^t) = \det(Q)\overline{\det(Q)} = |\det(Q)|^2$. Hence, $|\det(Q)| = 1$.

Problem 15. If A and B are similar, then $B = Q^{-1}AQ$ for some invertible Q. Hence

$$\det(B) = \det(Q^{-1}AQ) = \det(Q^{-1})\det(A)\det(Q) = \frac{1}{\det(Q)}\det(A)\det(Q) = \det(A).$$

Problem 17. Since $AB = -BA$, taking the determinant gives

$$\det(A)\det(B) = \det(AB) = \det(-BA) = (-1)^n\det(BA) = -\det(B)\det(A),$$

since n is odd. Thus, $2\det(A)\det(B) = 0$. Since $\mathrm{char}(F) \neq 2$, we have $\det(A)\det(B) = 0$. So $\det(A) = 0$ or $\det(B) = 0$, i.e. either A or B is not invertible.

Problem 22(c). The proof I have in mind uses the polynomial factorizations. For variables $x_0, x_1, \dots, x_n$, define

$$M(x_0, x_1, \dots, x_n) = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix},$$

and let $P(x_0, x_1, \dots, x_n)$ be the determinant of $M(x_0, \dots, x_n)$. Note that $M(x_0, \dots, x_n)$ is a matrix whose entries are polynomials! Similarly, $P(x_0, \dots, x_n)$ is a polynomial in the x_i's. By expansion along the rightmost column, we see that

$$\begin{aligned} P(x_0, \dots, x_n) =& (-1)^{n+1}x_0^n P(x_1, \dots, x_n) + (-1)^{n+2}x_1^n P(x_0, x_2, x_3, \dots, x_n) \\ &+ \dots + (-1)^{2n}x_n^n P(x_0, x_1, x_2, \dots, x_{n-1}). \end{aligned}$$

Hence, by induction, we can prove that $P(x_0,\ldots,x_n)$ is a homogeneous polynomial of degree $1+2+\cdots+n=\frac{n(n+1)}{2}$.

Now, suppose $0\le i<j\le n$ and $x_i=x_j$. Then the matrix $M(x_0,\ldots,x_n)$ would have two identical rows, and so its determinant $P(x_0,\ldots,x_n)=0$. In short, whenever $x_j-x_i=0$, we have $P(x_0,\ldots,x_n)=0$. By the factor theorem for polynomials, x_j-x_i is a factor of $P(x_0,\ldots,x_n)$. Thus we can write

$$P(x_0,\ldots,x_n)=Q(x_0,\ldots,x_n)\prod_{0\le i<j\le n}(x_j-x_i).$$

The left-hand side has degree $\frac{n(n+1)}{2}$ as we noted, and the right hand side has degree $\deg(Q)$ + number of pairs (i,j), $0\le i<j\le n$. This latter number is $1+2+\cdots+n=\frac{n(n+1)}{2}$, and so we get $\deg(Q)=0$, i.e. Q is constant.

To compute this constant, we note that the coefficient of $x_1^1x_2^2\ldots x_n^n$ in $P(x_0,x_1,\ldots,x_n)$ is 1. On the other hand, the corresponding coefficient in $\prod_{i<j}(x_j-x_i)$ is also 1 (since the only way to get $x_1^1x_2^2\ldots x_n^n$ in the product is to take x_n from x_n-x_0, $x_n-x_1,\ldots,x_n-x_{n-1}$; and x_{n-1} from $x_{n-1}-x_0$, $x_{n-1}-x_1,\ldots,x_{n-1}-x_{n-2}$ etc).

Problem 24. We wish to compute the determinant of:

$$B=A+tI=\begin{pmatrix} t & 0 & 0 & \ldots & 0 & a_0\\ -1 & t & 0 & \ldots & 0 & a_1\\ 0 & -1 & t & \ldots & 0 & a_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & -1 & a_{n-1}+t\end{pmatrix}.$$

Let $D(a_0,a_1,\ldots,a_{n-1})=\det(B)$. Expanding the first row, we get

$$\det(B)=(-1)^{n+1}a_0\det\begin{pmatrix} -1 & t & 0 & \ldots & 0\\ 0 & -1 & t & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & -1\end{pmatrix}+t\cdot\det\begin{pmatrix} t & 0 & 0 & \ldots & a_1\\ -1 & t & 0 & \ldots & a_2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & a_{n-1}+t\end{pmatrix}.$$

The first matrix has determinant $(-1)^{n-1}$, while the second has determinant $D(a_1,\ldots,a_{n-1})$. Hence, $D(a_0,a_1,\ldots,a_{n-1})=a_0+t\cdot D(a_1,\ldots,a_{n-1})$. Together with $D(a_{n-1})=a_{n-1}+t$, we get

$$D(a_0,\ldots,a_{n-1})=a_0+a_1t+a_2t^2+\cdots+a_{n-1}t^{n-1}+t^n.$$