MATH 110: LINEAR ALGEBRA
HOMEWORK #7

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§2.6: Dual Spaces

Problem 1. Note that "vector space" really means "finite-dim vector space" in these questions.

(a) F. The codomain of a linear functional must be the scalar field.
(b) T. The dimension of a field $\mathbb{F}$ considered as a vector space over itself is 1, so if $T : \mathbb{F} \to \mathbb{F}$ is linear, and β, γ are bases for $\mathbb{F}$, then $[T]_\beta^\gamma$ is $\dim(\mathbb{F}) * \dim(\mathbb{F})$, which is $1 * 1$.
(c) T. This is a corollary to Theorem 2.24: the basis β^* has the same number of elements as β. If V is infinite-dimensional, this may be false, but the proof of this fact isn't within the scope of the course.
(d) T/F. This depends on how you interpret "is". Certainly V is isomorphic to the dual of V^* (Theorem 2.26), so in that sense the statement is true. However V may not literally be a set of linear functionals, and in this sense, the statement is false. Again if V is inifinite-dimensional this can be false, and the proof of this is also outside the scope of the course.
(e) F. β is ordered, so if $\beta = \{e_1, \ldots, e_n\}$, $\beta^* = \{e_1^*, \ldots, e_n^*\}$ where e_i^* is the i^{th} coordinate projection functional, then $T(\beta) = \beta^*$ means $T(e_i) = e_i^*$ for each i, i.e. T must preserve the order of the basis elements. So by Theorem 2.6, there is only one linear T such that $T(\beta) = \beta^*$ (and this is an isomorphism). But if $n > 1$, to get extra ismorphisms between V and V^*, we could for example set $T(e_i) = e_{i+1}^*$ (and $T(e_n) = e_1^*$), and extend T to all of V linearly (by Theorem 2.6). We could also choose T so that $T(e_1) \notin \beta^*$, for example. If $n = 1$ and $\operatorname{char}(\mathbb{F}) \neq 2$, there are also multiple isomorphisms.
(f) T. $T^t : W^* \to V^*$ so $(T^t)^t : V^{**} \to W^{**}$.
(g) T. Let $\pi : V \to W$ be an isomorphism. Then π^t is an isomorphism between W^* and V^*. To see π^t is one-one, suppose $\pi^t(g_1) = \pi^t(g_2)$, so $g_1 \circ \pi = g_2 \circ \pi$, so g_1 and g_2 must agree (give the same outputs) on all of $\mathbf{Rg}(\pi)$, but π is onto, so $g_1 = g_2$. To see π_t is onto, let $f \in V^*$. Setting $g = f \circ \pi^{-1}$, as $\pi^{-1} : W \to V$, $g \in W^*$. Then $\pi^t(g) = f \circ \pi^{-1} \circ \pi = f$.
(h) F. The derivative of a function is another function, but the comdomain of a linear functional is the scalar field.

Problem 10. Here $V = \mathrm{P}_n(\mathbb{F})$ and $c_0, \ldots, c_n \in \mathbb{F}$ are all distinct.

(a) Firstly note that f_i is linear as $f_i(ap + q) = (ap + q)(c_i) = ap(c_i) + q(c_i)$. Also, the comdomain of f_i is $\mathbb{F}$, so $f_i \in V^*$. We want to see that $\{f_0, \ldots, f_n\}$ is an ordered

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basis for V^*. $\dim(V^*) = \dim(V) = n+1$ (by Theorem 2.24), so as long as all the f_i's are distinct, we have the right number of basis elements. So suppose

$$a_0 f_0 + \ldots + a_n f_n = 0.$$

Following the hint in the book, and generalizing, let

$$p(x) = (x - c_0)(x - c_1)\ldots(x - c_{i-1})(x - c_{i+1})\ldots(x - c_n). \tag{1}$$

Then

$$(a_0 f_0 + \ldots + a_n f_n)(p) = 0(p) = 0;$$

$$a_0 f_0(p) + \ldots + a_n f_n(p) = 0. \tag{2}$$

But plugging c_j's in for x in (1), $p(c_j) = 0$ if $j \neq i$, and

$$p(c_i) = (c_i - c_0)(c_i - c_1)\ldots(c_i - c_{i-1})(c_i - c_{i+1})\ldots(c_i - c_n).$$

The c_j's are all distinct, so $p(c_i)$ is the product of non-zero elements of $\mathbb{F}$, so $p(c_i) \neq 0$. But then (2) gives

$$0 + \ldots + 0 + a_i p(c_i) + 0 + \ldots + 0 = 0,$$

i.e. $a_i p(c_i) = 0$, so $a_i = 0$. Thus, for each i, $a_i = 0$.
Therefore the f_i's are all distinct (or it's easy to get a non-trivial linear combination), and are linearly independent, and so form a basis for V^*, by the earlier remarks on dimensions.

(b) Using the corollary to Theorem 2.26, let $\beta = \{p_0, \ldots, p_n\}$ be an ordered basis for V such that $\beta^* = \{f_0, \ldots, f_n\}$ is the dual basis to β. Then by definition of the dual basis, $f_j(p_i) = \delta_{ij}$, so $p_i(c_j) = \delta_{ij}$.
To see the p_i's are unique, suppose p_i^1 and p_i^2 have the property that $p_i^k(c_j) = \delta_{ij}$ for $k = 1, 2$. Let $p = p_i^1 - p_i^2$. Then clearly $f_j(p) = p(c_j) = 0$ for each j. But by (a), the f_j's span V^*, so $f(p) = 0$ for each $f \in V^*$, so by the lemma to Theorem 2.26, $p = 0$, and $p_i^1 = p_i^2$.

(c) As β (from (b)) forms a basis for V, we have $V = \mathrm{span}(\beta)$. If $b_i \in \mathbb{F}$, then

$$(\sum_{i=0}^{i=n} b_i p_i)(c_j) = \sum_{i=0}^{i=n} b_i p_i(c_j) = \sum_{i=0}^{i=n} b_i \delta_{ij} = b_j. \tag{3}$$

So if we set $q = \sum_{i=0}^{i=n} a_i p_i$, $q(c_j) = a_j$ follows. The uniqueness of q with this property also follows, as if $p \neq q$ then p is not expressed by the same linear combination of β as q is.

(d) Let $p \in V$. Let $b_i \in \mathbb{F}$ be such that $p = \sum_{i=0}^{i=n} b_i p_i$. Then by (3), $p(c_j) = b_j$, so in fact, $p = \sum_{i=0}^{i=n} p(c_i) p_i$.

(e) Using (d), $\int_a^b p(t)dt =$

$$= \int_a^b \left(\sum_{i=0}^{i=n} p(c_i) p_i\right)(t)\ dt = \int_a^b \sum_{i=0}^{i=n} p(c_i) p_i(t)\ dt$$

$$= \sum_{i=0}^{i=n} \left(p(c_i) \int_a^b p_i(t)\ dt\right) = \sum_{i=0}^{i=n} p(c_i) d_i,$$

where $d_i = \int_a^b p_i(t)\, dt$. Note the linearity of integration has been used, and note that the $p(c_i)$'s are just scalars, so they are pulled outside the integral.
Suppose $c_i = a + (b-a)\frac{i}{n}$ and $a < b$. If $n = 1$, then setting $p_0(x) = \frac{b-x}{b-a}$ and $p_1(x) = \frac{x-a}{b-a}$, then p_0, p_1 are the unique polynomials (of degree ≤ 1) of part (b). Then we get $d_0 = d_1 = \frac{b-a}{2}$, and substituting this in (e) gives the trapezoidal rule. The $n = 2$ case is similar.

Matrices

For several problems in this section there are various possible solutions, so in several cases I give a couple of these.

Problem 2.

Lower triangular.

Let A and B be lower triangular, $m * n$ and $n * p$ respectively. Let $L = AB$, so L is $m * p$. We want to see that L is lower triangular.

Solution 1:

We can write each of A and B as $2 * 2$ block matrices, and calculate $L = AB$ using the lemma covered in class on multiplication of block matrices (also generalized in problem 10 of this homework). We need to see that if $k < j$ then $L_{kj} = 0$. But notice that if $k < j$ and we write L as a block matrix

$$L = \begin{bmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{bmatrix}.$$

where L^{11} is $k * k$, the $(k, j)^{\text{th}}$ entry of L is within L^{12}, and L^{12} lies above the main diagonal. (Notice if $k = m$ then the partition would actually be $1 * 2$. I'll assume $k < m$ as this is the more complicated case. It's easy to adapt the following argument to the $k = m$ case, by setting all partitions of A to be $1 * x$.) So if we can show that $L^{12} = 0$, we will have proven L is lower triangular. We will do this.

We want to partition A and B into block matrices so that, block-multiplying them, we produce L, with the given partition. There are two cases:

Case 1: $k < n$.

Partition A and B into $2 * 2$ block matrices with submatrices A^{ij}, B^{ij}, where A^{11} and B^{11} are both $k * k$ (therefore A^{21} is $m - k * k$, etc). (Note B is $n * p$ and $k < p$ so this partition will be $2 * 2$ for B). It's easy to check that because AB makes sense, block-multiplication makes sense with these partitions (if you've printed this out you should write the dimensions on the diagram below to check it all works; I can't work out how to do it with good alignment). Moreover, the partition induced on L by block-multiplication agrees with the above partition on L. Because A^{11} and B^{11} are square and A and B are lower triangular, we have $A^{12} = B^{12} = 0$. So block-multiplying, we have

$$\begin{bmatrix} A^{11} & 0 \\ A^{21} & A^{22} \end{bmatrix} \begin{bmatrix} B^{11} & 0 \\ B^{21} & B^{22} \end{bmatrix} = \begin{bmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{bmatrix},$$

so

$$L^{12} = A^{11}0 + 0B^{22} = 0.$$

Case 2: $k \geq n$.

In this case we will partition A as a $2 * 1$ block matrix where A^{11} is $k * n$ and B as a $1 * 2$

block matrix with B^{11} $n*k$. Again this makes sense for block-multiplication and agrees with the established partition of L. As $n \geq k$ we have $B^{12} = 0$, so we get $L^{12} = A^{11}0 = 0$, as required.
So we have shown what we needed, and therefore L is lower triangular.

Solution 2:
Let $i < j$; we just need $(AB)_{ij} = 0$. But

$$(AB)_{ij} = \sum_{k=1}^{k=n} A_{ik}B_{kj} = \sum_{k=1}^{k=i} A_{ik}B_{kj} + \sum_{k=i+1}^{k=n} A_{ik}B_{kj}.$$

In the last expression, the left summand is 0 because if $k \leq i$ then $k < j$, so $B_{kj} = 0$ as B is lower triangular. Similarly, the right summand is 0 because if $i+1 \leq k$, then $A_{ik} = 0$ because $i < k$ and A is lower triangular.

Upper triangular. If A and B are upper triangular, then $(AB)^t = B^tA^t$ and B^t and A^t are lower triangular, so by the first part $(AB)^t$ is lower triangular, so AB is upper triangular.

Problem 3.
Lower triangular.
Let A be lower triangular and invertible, $n*n$.

Solution 1:
Let $a < b$. We need to see $A^{-1}_{ab} = 0$. Consider the partition of $B = A^{-1}$ into a $2*2$ block matrix with partition dimensions $n_j * p_k$ where $n_1 = p_1 = a$ (so $n_2 = p_2 = n - a$). Note that the $(a,b)^{\text{th}}$ entry of B lies in B^{12}, so we will be done if we can show $B^{12} = 0$.
Let A also be partitioned as $2*2$, with the same partition dimensions. Note that as $n_1 = p_1$, A^{11} and B^{11} are square and A^{12} (and B^{12}) lies above the diagonal, so $A^{12} = 0$. Moreover, the partitions are suitable for block-multiplying, giving

$$\begin{bmatrix} A^{11} & 0 \\ A^{21} & A^{22} \end{bmatrix} \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} = \begin{bmatrix} I_a & 0 \\ 0 & I_{n-a} \end{bmatrix}.$$

The partitioning of $AB = I_n$ is again the same, and the upper-left and lower-right blocks are square, $a*a$ and $(n-a)*(n-a)$ respectively, so it clearly has the form shown. But this gives

$$A^{11}B^{11} + 0B^{21} = I_a$$
$$\implies A^{11}B^{11} = I_a.$$

But then by problem 2.4.10 from homework 6, this implies A^{11} is invertible. (One could also use the fact that a square lower triangular matrix with non-zero diagonal entries is invertible.) We also get

$$A^{11}B^{12} + 0B^{22} = 0 \implies A^{11}B^{12} = 0.$$

But then left-multiplying by $(A^{11})^{-1}$, we get $B^{12} = 0$.

Solution 2:
Suppose $k < j$ are such that $A^{-1}_{kj} \neq 0$. We may assume that $A^{-1}_{ij} = 0$ for $i < k$ (by reducing

k if necessary). But then

$$(AA^{-1})_{kj} = \sum_{i=1}^{i=n} A_{ki}A^{-1}_{ij} = \sum_{i<k} A_{ki}A^{-1}_{ij} + A_{kk}A^{-1}_{kj} + \sum_{i>k} A_{ki}A^{-1}_{ij}.$$

In the last term, the left summand is 0 because $A^{-1}_{ij} = 0$ for $i < k$. The right summand is 0 because A is lower triangular. But as A is also invertible, $A_{kk} \neq 0$. But then $(AA^{-1})_{kj} \neq 0$, contradicting $I_{kj} = 0$.

Solution 3:
Let $\beta = \{e_1, \ldots, e_n\}$ be the standard ordered basis for $\mathbb{F}^n$. Note first that because $L_A(e_j) = \sum_{i=1}^{i=n} A_{ij}e_i$ and $A_{ij} = 0$ for $i < j$, we actually have,

$$L_A(e_j) = \sum_{i=j}^{i=n} A_{ij}e_i; \quad L_A(e_j) \in \operatorname{span}(e_j, \ldots, e_n). \tag{4}$$

Then letting $W_j = \operatorname{span}(e_j, \ldots, e_n)$, it's easy to see that W_j is L_A-invariant for each j. Assuming A^{-1} is not lower triangular, choose k, j with $k < j$ as in solution 2, and let $a = A^{-1}_{kj} \neq 0$. Then

$$L_{A^{-1}}(e_j) = \sum_{i=1}^{i=n} A^{-1}_{ij}e_i = ae_k + \sum_{i=k+1}^{i=n} A^{-1}_{ij}e_i$$

by the choice of k. Let $w = \sum_{i=k+1}^{i=n} A^{-1}_{ij}e_i$, noting that $w \in W_{k+1}$. Then

$$e_j = L_A(L_{A^{-1}}(e_j)) = L_A(ae_k + w) = aL_A(e_k) + L_A(w),$$

where the first equality is because $AA^{-1} = I_n$. As $w \in W_{k+1}$, so is $L_A(w)$, by L_A-invariance. Using (4) we have $L_A(e_k) = A_{kk}e_k + v$ for some $v \in W_{k+1}$. Also $e_j \in W_{k+1}$ as $j \geq k+1$. So

$$e_j - L_A(w) - av = aA_{kk}e_k.$$

But $a \neq 0$ by assumption and $A_{kk} \neq 0$ as A is lower triangular and invertible, so we can divide through and get $e_k \in W_{k+1} = \operatorname{span}(e_{k+1}, \ldots, e_n)$, a contradiction, as β is a basis.

Upper triangular. Finally, let A be upper triangular and invertible. By problem 5 of section 2.4, A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$. But A^t is also lower triangular, so by the previous part, $(A^t)^{-1}$ is lower triangular. So $A^{-1} = ((A^{-1})^t)^t = ((A^t)^{-1})^t$ is upper triangular.

Problem 4. Let A and B be unit lower triangular matrices, $m * n$ and $n * p$. By problem 2 we need only check that $C = AB$ is also unit.

Solution 1:
If $n = 1$, A is a column vector and B is a row of the form $[10 \ldots 0]$. Computing the product directly, C's only non-zero column is its first column, and has the same entries as A. Therefore it is a unit matrix. If $m = 1$ or $p = 1$ it is also easy to check that C is unit. So assume $m, n, p > 1$. We use induction on $\max(m, n, p)$.
Partition A into a $2 * 2$ block matrix A' where A'_{11} is $1 * 1$ (so the partition dimensions are given by $m_1 = 1$, $m_2 = m - 1$, $n_1 = 1$ and $n_2 = n - 1$). Define B' from B in the same way,

so that B'_{11} is also $1*1$. (All the dimensions are >0 as $m,n,p>1$.) Block-multiplying, we produce a $2*2$ partition C' of C:

$$C' = \begin{bmatrix} [1] & 0 \\ A'_{21} & A'_{22} \end{bmatrix} \begin{bmatrix} [1] & 0 \\ B'_{21} & B'_{22} \end{bmatrix}.$$

(A' and B' have this form because A and B are unit lower trianuglar.) But then $C'_{11} = [1][1] + 0B'_{21} = [1]$, so $C_{11} = 1$. And $C'_{22} = A'_{21}0 + A'_{22}B'_{22} = A'_{22}B'_{22}$. But this is a product of unit lower triangular matrices of smaller dimensions, so by inductive hypothesis, C'_{22} is also unit. As all of C's diagonal entries are either $C_{11} = 1$ or lie within C'_{22}, C must be unit.

Solution 2:
More directly, just calculate $(AB)_{ii}$:

$$(AB)_{ii} = \sum_{k=1}^{k=n} A_{ik}B_{ki} = \sum_{k=1}^{k=i-1} A_{ik}B_{ki} + A_{ii}B_{ii} + \sum_{k=i+1}^{k=n} A_{ik}B_{ki}. \tag{5}$$

In the last term, the left summand is 0 because $B_{ki} = 0$ for $k < i$. The right summand is 0 because $A_{ik} = 0$ for $k > i$. But $A_{ii} = B_{ii} = 1$ as A and B are unit matrices, so $(AB)_{ii} = 1.1 = 1$ also.

Upper triangular. If A and B are unit upper triangular then, like before, $(AB)^t = B^tA^t$ is unit lower triangular, so AB is unit upper triangular.

Problem 5.
Lower triangular. Let A be a unit lower triangular square matrix. As all diagonal entries are non-zero, A is invertible. By problem 3 we already know A^{-1} is lower triangular.

Solution 1:
Use induction on the dimensions of A.
If A is $1*1$, $A = [1]$ as it is unit, so clearly $A^{-1} = [1]$ also.
Now suppose A is $(n+1)*(n+1)$ where $n > 0$. Partition A into a $2*2$ block matrix A' with A'_{11} of size $1*1$, and define B' from B in the same way. Block-multiplying, we get the same partition of I_{n+1}:

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} [1] & 0 \\ A'_{21} & A'_{22} \end{bmatrix} \begin{bmatrix} B'_{11} & 0 \\ B'_{21} & B'_{22} \end{bmatrix}.$$

($B'_{12} = 0$ because we already know it is lower triangular.) But then $I_1 = [1]B'_{11} + 0B'_{21} = [1]B'_{11}$, and therefore $B'_{11} = [1]$, so $B_{11} = 1$.
Now A'_{22} is unit lower triangular square (so invertible), because A is. Moreover, its inverse is B'_{22}, because, from the matrix equation,

$$I_n = A'_{21}0 + A'_{22}B'_{22} = A'_{22}B'_{22}.$$

As A'_{22} is $n*n$, we may apply the inductive hypothesis, so B'_{22} is unit. Now we've dealt with all of B's diagonal entries, so B is unit also.

Solution 2:
Let $B = A^{-1}$, and consider (5). By problem 3, A^{-1} is lower triangular, so again the left and right summands of the last term are 0. Therefore $1 = (AA^{-1})_{ii} = A_{ii}A^{-1}_{ii}$ (it's 1 because

$AA^{-1} = I$), and $A_{ii} = 1$, so $A_{ii}^{-1} = 1$ also.

Upper triangular. If A is a unit upper triangular square matrix, then applying the lower triangular case to A^t, we get $(A^t)^{-1}$ is unit, and therefore $A^{-1} = ((A^t)^{-1})^t$ is unit also.

Problem 10. Let A be $m * n$ and B be $n * p$ matrices. Suppose

$$m = m_1 + \ldots + m_r$$

$$n = n_1 + \ldots + n_s$$

$$p = p_1 + \ldots + p_t.$$

I'll distinguish here between a matrix and a block-representation by giving them different names. We can form an $r * s$ block matrix A^* representing A:

$$A^* = \begin{bmatrix} A^*_{11} & A^*_{12} & \ldots & A^*_{1s} \\ A^*_{21} & A^*_{22} & \ldots & A^*_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A^*_{r1} & A^*_{r2} & \ldots & A^*_{rs} \end{bmatrix},$$

where A^*_{ij} is an $m_i * n_j$ (regular) matrix with coefficients matching the section of A it corresponds to. More precisely, define $(A^*_{ij})_{ab} = A_{(\sum_{k<i} m_i + a)(\sum_{k<j} n_i + b)}$. For $i \leq r+1$, set $m_{<i} = \sum_{k<i} m_i$, and similarly define $n_{<j}$ and $p_{<k}$. Then we have

$$(A^*_{ij})_{ab} = A_{(m_{<i}+a)(n_{<j}+b)}.$$

We can represent B similarly, forming an $s*t$ block matrix B^*, where block B^*_{jk} is an $n_j * p_k$ matrix. Then the multiplication $A^*_{ij}B^*_{jk}$ makes sense, and yields an $m_i * p_k$ matrix. This holds for any j, so we can define an $r * t$ block matrix D^* by $D^*_{ik} = \sum_{j=1}^{j=s} A^*_{ij}B^*_{jk}$. Let D be the corresponding $m * p$ (regular) matrix.
Note that the way I have defined things, A^* is actually different from A, it is not just a different way of representing A. The dimensions of A^* are $r * s$, not $m * n$, and its entries are matrices, not scalars, as with A. Likewise for B^*, and the product A^*B^*. It is most convenient for this proof to view things this way.
On the other hand, letting $C = AB$ (with regular matrix multiplication), C is $m * p$, and we can form the $r * t$ block matrix C^*, where block C^*_{ik} is $m_i * p_k$ (as we did for A and B). So we have that the two block matrices C^* and D^* have the same dimensions. We need to verify that $C^*_{ik} = D^*_{ik}$. One can do this inductively, but here I'll just do it directly.
To make things more readable, I'll move some subscripts to superscipts. For X and X^* any of the matrices and corresponding block matrices above, let $X^{ij} = X^*_{ij}$. So X^{ij} is a submatrix of X. However X_{ij}, as usual, is the $(i, j)^{\text{th}}$ entry of X.
Now, we need to see that $C^*_{ik} = D^*_{ik}$. Firstly, each is $m_i * p_k$. So let $1 \leq a \leq m_i$ and $1 \leq c \leq p_k$. We just need to check that $C^{ik}_{ac} = D^{ik}_{ac}$. Computing,

$$D^{ik}_{ac} = (\sum_{j=1}^{j=s} A^{ij}B^{jk})_{ac} = \sum_{j=1}^{j=s} (A^{ij}B^{jk})_{ac} =$$

$$= \sum_{j=1}^{j=s} \sum_{b=1}^{b=n_j} (A^{ij}_{ab}B^{jk}_{bc})$$

$$= \sum_{j=1}^{j=s} \sum_{b=1}^{b=n_j} A_{(m_{<i}+a)(n_{<j}+b)} B_{(n_{<j}+b)(p_{<k}+c)}$$

$$= \sum_{j=1}^{j=s} \sum_{b=1}^{b=n_j} A_{a'(n_{<j}+b)} B_{(n_{<j}+b)c'}$$

(where $a' = m_{<i} + a$ and $c' = p_{<k} + c$)

$$= \sum_{j=1}^{j=s} \sum_{b=(n_{<j})+1}^{b=n_{<(j+1)}} A_{a'b} B_{bc'},$$

as $n_{<j} + n_j = n_{<(j+1)}$. But this is just

$$= \sum_{b=1}^{b=n} A_{a'b} B_{bc'} = C_{a'c'} = C_{(m_{<i}+a)(p_{<k}+c)} = C^{ij}_{ac},$$

as required.

Note: Proving this inductively avoids the heavy computation done here, but still seems to need a fair bit of notation. I originally wrote an inductive proof also, but it became too involved to be worth including in the solutions, and appeared more complicated than it really is. But the idea is fairly straightforward.
Suppose we are given some block matrices A^* and B^* partitioning matrices A and B. We want to prove that if we compute the product A^*B^* at the partition level, then throw away the partition on the resulting matrix, that we get the regular product AB. To use induction, we need to break the problem into a few smaller ones. This can be done by partitioning the block matrices A^* and B^* into $2 * 2$ matrices $A^{*\boxplus}$ and $B^{*\boxplus}$. These matrices are another level up - their entries are block matrices (pieces of A^* and B^*), whose entries are (regular) matrices. But the $2 * 2$ base case still applies to these "nested" matrices. To multiply $A^{*\boxplus}$ with $B^{*\boxplus}$ we need to multiply their block-matrix components, but these are smaller than the ones we started with (A' and B'), so we can use the inductive hypothesis to conclude that the result is the same as multiplying the corresponding pieces of A and B. To write all this carefully seems to need a fair bit of notation, which could well cause the reader to lose the ideas amongst the symbols. So if you're interested in how this proof works, I suggest you think about the details on your own.