MATH 110: LINEAR ALGEBRA
HOMEWORK #6

Problem (1). By the corollary on page 7 of the lecture notes from September 28, for each i and j (with $1 \leq i, j \leq n$ and $i \neq j$), B_{ij} is the number of paths of length at most $n-1$ from i to j. So if $B_{ij} = 0$, then there are no paths of length $n-1$ or less from i to j, i.e., $(A^m)_{ij} = 0$ for $1 \leq m \leq n-1$. What remains to be shown is that in this situation there are no paths from i to j of length greater than $n-1$. We will prove the contrapositive of this statement.

Claim. Let G be a graph with n vertices. If i, j are distinct vertices such that there is a path from i to j, then there is such a path of length $\leq n-1$.

Proof. We proceed by induction on n. If $n = 1$, then there is only one distinct vertex, so the statement is automatically true. Now, assume that the statement holds for a graph with n vertices. We want to show that it holds for a graph with $n+1$ vertices. Let i and j be two distinct vertices in such a graph, and assume that there is a path from i to j. Let P be the shortest path from i to j, and let $p_0, \dots, p_m$ be the vertices that that occur along this path, listed in the order they occur as one travels from i to j. (So, in particular, $p_0 = i$ and $p_m = j$.) If $m = 1$, then we're done, since we then have a path of length 1. Otherwise, consider the graph G' obtained from G by removing the vertex j and the edges that are connected to j. Since P is the shortest path from i to j, the vertices $p_0, \dots, p_{m-1}$ are all different from j, and thus lie in G'. Also, there is a path from $i = p_0$ to p_{m-1}, so, by the inductive hypothesis, there is a path from i to p_{m-1} of length $\leq n-1$. Hence there is a path from i to j in G of length $\leq n$, since p_{m-1} is connected to j by an edge.

Problem (2). $C = BR$. To see this, note that for all i and j with $1 \leq i, j \leq n$, $(BR)_{ij} = \sum_{k=1}^{n} B_{ik}R_{kj}$ is equal to the number of positive integers k (with $1 \leq k \leq n$) such that $B_{ik} = 1$ and $R_{kj} = 1$ (since the only possible values of the B_{ik}'s and the R_{kj}'s are 0 and 1). This, in turn, is precisely the number of paths from i to j consisting of one blue edge followed by one red edge, since $B_{ik} = 1$ iff there is a blue edge from i to k, and $R_{kj} = 1$ iff there is a red edge from k to j.

By a similar argument, $D = RB$.

§2.3: Composition of Linear Transformations and Matrix Multiplication

Problem 15. Let $v_1, \dots, v_n$ denote the columns of A. By Theorem 2.3, the ith column of MA is Mv_i, for $i \in \{1, \dots, n\}$. Now, suppose that $v_j = a_1v_1 + \cdots + a_nv_n$ for some $j \in \{1, \dots, n\}$ and $a_1, \dots, a_n \in F$ (some of which may be zero). Then $Mv_j = M(a_1v_1 + \cdots + a_nv_n) = a_1Mv_1 + \cdots + a_nMv_n$, as desired.

§2.4: Invertibility and Isomorphisms

Problem 1.

(a) False. The expression $[T^{-1}]_\alpha^\beta$ doesn't quite make sense, since T^{-1} is a linear transformation from W to V, while α is a basis for V and β is a basis for W.

(b) True. See page 100.

(c) False. T is a linear transformation from V to W, whereas L_A is a linear transformation from F^n to F^m (where $n = \dim(V)$ and $m = \dim(W)$). The correct statement is $\phi_\beta T = L_A \phi_\alpha$, using the notation on page 104.
(d) False, by Theorem 2.19, since $\dim(\mathbb{M}_{2\times 3}(F)) = 6 \neq 5 = \dim(F^5)$.
(e) True, by Theorem 2.19.
(f) False. For example, let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $AB = I_2$, but neither A nor B is invertible (they are not even square).
(g) True. Note that $AA^{-1} = A^{-1}A = I$ and $(A^{-1})^{-1}A^{-1} = A^{-1}(A^{-1})^{-1} = I$. So we conclude that $A = (A^{-1})^{-1}$ from the fact that inverses of matrices are unique (see bottom of page 100).
(h) True. See Corollary 2 on page 102.
(i) True. See the definition of invertible matrix.

Problem 4. $(AB)(B^{-1}A^{-1}) = (A(BB^{-1}))A^{-1} = (AI)A^{-1} = AA^{-1} = I$, and $(B^{-1}A^{-1})(AB) = (B^{-1}(A^{-1}A))B = (B^{-1}I)B = B^{-1}B = I$, by definition of inverse. Hence $B^{-1}A^{-1}$ is the inverse of AB, by the same definition (i.e., $(AB)^{-1} = B^{-1}A^{-1}$).

Problem 5. By the computation on page 88, for any composable matrices A and B, $(AB)^t = B^tA^t$. So, setting $B = A^{-1}$, we conclude that $A^t(A^{-1})^t = (A^{-1}A)^t = I^t = I$ and $(A^{-1})^tA^t = (AA^{-1})^t = I^t = I$. Hence $(A^{-1})^t$ is the inverse of A^t, i.e., $(A^t)^{-1} = (A^{-1})^t$.

Problem 9. By Theorem 2.15, $L_AL_B = L_{AB}$. Since AB is invertible, so is L_{AB}, by Corollary 2 on page 102 (and, hence, so is L_AL_B). In particular, L_AL_B is one-to-one and onto. So, by Exercise 2.3.12, L_B is one-to-one and L_A is onto. By Theorem 2.5, $\mathrm{rank(L_A)} = \dim(F^n) = \mathrm{rank(L_B)}$. So L_A and L_B are invertible, by Remark 3 on page 100. Hence, A and B are invertible, by Corollary 2 on page 102.

For the second part of the problem we can use the same example as in Problem 1(f) above.

Problem 10.

(a) I_n is invertible, so A and B are also invertible, by the previous problem.
(b) Let $C = B^{-1}$. Then $C = I_nC = (AB)C = A(BC) = AI_n = A$. So $A = B^{-1}$.
(c) **Claim.** Let V and W be n-dimensional vector spaces, and let $S : V \to W$ and $T : W \to V$ be linear transformations such that $ST = I_W$. Then S and T are invertible, and $S^{-1} = T$.

Proof. Let α and β be ordered bases for V and W, respectively. Then $[ST]_\beta = [S]_\alpha^\beta[T]_\beta^\alpha$, by Theorem 2.11. Also, $[I_W]_\beta = I_n$, by Theorem 2.12. Hence, $[S]_\alpha^\beta[T]_\beta^\alpha = I_n$. By (a) and (b), $[S]_\alpha^\beta$ and $[T]_\beta^\alpha$ are invertible, and $[S]_\alpha^\beta = ([T]_\beta^\alpha)^{-1}$. By Theorem 2.18, S and T are invertible, and $[T^{-1}]_\alpha^\beta = ([T]_\beta^\alpha)^{-1}$. So $[S]_\alpha^\beta = [T^{-1}]_\alpha^\beta$, and hence $S^{-1} = T$, by the remark at the bottom of page 80.

§2.5: The Change of Coordinate Matrix

Problem 1.

(a) False. Let β, β', and Q be as in Example 1 on page 112. Then the first column of Q is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, while $[x_1]_{\beta'} = [(1,1)]_{\beta'} = \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix}$.

(b) True, by Theorem 2.22.
(c) True. By Theorem 2.23, $[T]_{\beta'} = Q^{-1}[T]_\beta Q$. Multiplying both sides by Q on the left and Q^{-1} on the right, we get $Q[T]_{\beta'}Q^{-1} = [T]_\beta$.
(d) False. See the definition of similarity on page 115.
(e) True, by Theorem 2.23.

Problem 6.

(a) For any vector $(x, y) \in F^2$, $L_A(x, y) = (x+3y, x+y)$. Applying L_A to the vectors in β, we have $L_A(1, 1) = (4, 2) = 6(1, 1) - 2(1, 2)$ and $L_A(1, 2) = (7, 3) = 11(1, 1) - 4(1, 2)$, where the coefficients are obtained by solving the linear systems

$$4 = a * 1 + b * 1$$
$$2 = a * 1 + b * 2$$

and

$$7 = a * 1 + b * 1$$
$$3 = a * 1 + b * 2,$$

respectively. Hence, $[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$. By the Corollary on page 115, if $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (where the columns correspond to the vectors in β), then $[L_A]_\beta = Q^{-1}AQ$.

(b) For the sake of variety, instead of going through the direct computation, as we did in the previous part, let's do this part using $[L_A]_\beta = Q^{-1}AQ$. Set $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Using techniques from Math 54, let us compute Q^{-1}:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right) \to$$
$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right),$$

where we first subtract the first row from the second, then divide the second row by -2, and then subtract the second row from the first. Thus, $Q^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$, and so $[L_A]_\beta = Q^{-1}AQ = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

Problem 10. Since A and B are similar, there is an invertible $n \times n$ matrix Q such that $B = Q^{-1}AQ$. By Exercise 2.3.13, $\mathrm{tr}(B) = \mathrm{tr}((Q^{-1}A)Q) = \mathrm{tr}(Q(Q^{-1}A)) = \mathrm{tr}((QQ^{-1})A) = \mathrm{tr}(IA) = \mathrm{tr}(A)$. So $\mathrm{tr}(B) = \mathrm{tr}(A)$, as desired.