MATH 110: LINEAR ALGEBRA
HOMEWORK #2

§1.5: Linear Dependence and Linear Independence

Problem 1.

(a) False. The set $\{(1,0),(0,1),(0,-1)\}$ is linearly dependent but $(1,0)$ is not a linear combination of the other 2 vectors.
(b) True. If 0_V is in the set, then $1 \cdot 0_V = 0_V$ is a nontrivial linear relation.
(c) False. Without any vectors in the set, we cannot form any linear relations.
(d) False. Take the set in (a), and look at the subset $\{(1,0),(0,1)\}$.
(e) True, by corollary to theorem 1.6 on page 39.
(f) True. This is precisely the definition of linear independence.

Problem 2.

(b) Linearly independent. For suppose that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = a \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} + b \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} a-b & -2a+b \\ -a+2b & 4a-4b \end{pmatrix}$$

for some $a, b \in \mathbb{R}$. The corresponding four linear equations are $0 = a-b$, $0 = -2a+b$, $0 = -a+2b$, and $0 = 4a-4b$. By the first equation, $a = b$, and so, by the second equation, $0 = -b$. Therefore, $a = b = 0$.
(d) Linearly dependent, because $-2(x^3-x)+(3/2)(2x^2+4)-1(-2x^3+3x^2+2x+6) = -2x^3+2x+3x^2+6-(-2x^3+3x^2+2x+6) = 0$.

Problem 8.

(a) Suppose that $a(1,1,0)+b(1,0,1)+c(0,1,1) = 0$ for some $a, b, c \in \mathbb{R}$. Then $(a+b, a+c, b+c) = 0$, and so a, b, and c must satisfy the following system of linear equations: $a+b = 0$, $a+c = 0$, $b+c = 0$. Subtracting the second equation from the first, we obtain $b-c = 0$. Adding this to the third equation, we see that $2b = 0$. Multiplying this equation by $1/2$ on both sides, we obtain $b = 0$. Consequently, $a = c = 0$ as well. Hence S must be linearly independent.
(b) If F has characteristic 2, then we can no longer multiply the above equation $2b = 0$ by $1/2$ to conclude that $b = 0$. In fact, in this case, $2b = 0$ is satisfied by all $b \in F$. So if we let b be any nonzero element of F and take $a = c = -b$ $(= b)$, we will have a solution to $a(1,1,0)+b(1,0,1)+c(0,1,1) = 0$ where the scalars are nonzero. In particular, $1(1,1,0)+1(1,0,1)+1(0,1,1) = 0$, and so S is linearly dependent.

Problem 9. Suppose that $\{u, v\}$ is linearly dependent. Then $au+bv = 0$ for some $a, b \in F$, where at least one of a and b is nonzero. Without loss of generality, we may assume that $a \neq 0$. Since F is a field, a must have a multiplicative inverse, $a^{-1} \in F$. So $u = (-a^{-1}b)v$, i.e., u is a multiple of v.

Conversely, suppose that u or v is a multiple of the other. Without loss of generality, we may assume that u is a multiple of v, that is $u = av$ for some $a \in F$. Hence, $1u - av = 0$, and so $\{u, v\}$ is linearly dependent, by definition of linear dependence.

Problem 12. For theorem 1.6, suppose S_1 is linearly dependent. Hence we can find elements $s_1, s_2, \ldots, s_n \in S_1$, and scalars $c_1, \ldots, c_n$ (not all 0), such that

$$c_1 s_1 + c_2 s_2 + \cdots + c_n s_n = 0.$$

Since $S_1 \subseteq S_2$, each s_i is also an element of S_2. Thus, the above linear relation in S_1 gives us a linear relation in S_2, and we see that S_2 is linearly dependent.

The corollary follows since it is precisely the contrapositive statement of theorem 1.6. *In other words, saying $A \implies B$ is the same as saying $\neg B \implies \neg A$.*

Problem 13.

(a) Suppose that $\{u, v\}$ is linearly independent and that $a(u+v)+b(u-v) = 0$ for some $a, b \in F$. Then $(a + b)u + (a - b)v = 0$. Since $\{u, v\}$ is linearly independent, this means that $a + b = 0$ and $a - b = 0$. Adding these two equations together, we get $2a = 0$. Since the characteristic of F is not equal to two, we conclude that $a = 0$. Consequently, $b = 0$ as well. Hence $\{u + v, u - v\}$ is linearly independent.

Conversely, suppose that $\{u + v, u - v\}$ is linearly independent. Let $s = u + v$ and $t = u - v$. Since $\{s, t\}$ is linearly independent, by the previous paragraph, $\{s+t, s-t\}$ is linearly independent. But, $s+t = 2u$ and $s-t = 2v$. Now, the fact that $\{2u, 2v\}$ is linearly independent implies that $\{u, v\}$ is linearly independent. (Since if $au+bv = 0$ for some $a, b \in F$, then $2au + 2bv = a(2u) + b(2v) = 0$, implying that $a = b = 0$.)

(b) Suppose that $\{u, v, w\}$ is linearly independent and that $a(u+v)+b(u+w)+c(v+w) = 0$ for some $a, b, c \in F$. Then $(a+b)u+(a+c)v+(b+c)w = 0$. Since $\{u, v, w\}$ is linearly independent, this means that $a + b = 0$, $a + c = 0$, and $b + c = 0$. Subtracting the second equation from the first, we get $b - c = 0$. Adding this to the third equation, we get $2b = 0$. Since the characteristic of F is not equal to two, we conclude that $b = 0$. Consequently, $a = c = 0$ as well. Hence $\{u + v, u + w, v + w\}$ is linearly independent.

Conversely, suppose that $\{u + v, u + w, v + w\}$ is linearly independent. Note that $u = \frac{1}{2}(u + v) + \frac{1}{2}(u + w) - \frac{1}{2}(v + w)$, $v = \frac{1}{2}(u + v) - \frac{1}{2}(u + w) + \frac{1}{2}(v + w)$, and $w = -\frac{1}{2}(u + v) + \frac{1}{2}(u + w) + \frac{1}{2}(v + w)$. (We can divide by 2, since the characteristic of F is not equal to 2.) Suppose that $au + bv + cw = 0$ for some $a, b, c \in F$. Then $a(\frac{1}{2}(u+v)+\frac{1}{2}(u+w)-\frac{1}{2}(v+w))+b(\frac{1}{2}(u+v)-\frac{1}{2}(u+w)+\frac{1}{2}(v+w))+c(-\frac{1}{2}(u+v)+\frac{1}{2}(u+w)+\frac{1}{2}(v+w)) = 0$, and so $\frac{a+b-c}{2}(u+v)+\frac{a-b+c}{2}(u+w)+\frac{-a+b+c}{2}(v+w) = 0$. Since $\{u+v, u+w, v+w\}$ is linearly independent, we have that $\frac{a+b-c}{2} = 0$, $\frac{a-b+c}{2} = 0$, and $\frac{-a+b+c}{2} = 0$. Clearly, the only solution to this system of equations is $a = b = c = 0$. So $\{u, v, w\}$ is linearly independent.

Problem 17. If M is an $n \times n$ upper triangular matrix with nonzero entries, then

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & a_{22} & a_{23} & \ldots & a_{2n} \\ 0 & 0 & a_{33} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{pmatrix},$$

where each $a_{ij} \in F$, and $a_{11}, a_{22}, \ldots, a_{nn}$ are not zero. To show that the columns are linearly independent, suppose that

$$r_1 \begin{pmatrix} a_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + r_2 \begin{pmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + r_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for some $r_1, r_2, \ldots, r_n \in F$. This leads to a system of n linear equations:
$r_1a_{11} + r_2a_{12} + r_3a_{13} + \cdots + r_na_{1n} = 0$
$0 + r_2a_{22} + r_3a_{23} + \cdots + r_na_{2n} = 0$
$0 + 0 + r_3a_{33} + \cdots + r_na_{3n} = 0$
$\vdots$
$0 + \cdots + 0 + r_{n-1}a_{n-1,n-1} + r_na_{n-1,n} = 0$
$0 + \cdots + 0 + 0 + r_na_{nn} = 0.$
Now, since $a_{nn} \neq 0$, the last equations implies that $r_n = 0$. Substituting this into the next to the last equation, we see that $r_{n-1} = 0$. Continuing in this fashion, we conclude that $r_1 = r_2 = \cdots = r_n = 0$. Hence the columns are linearly independent.

§1.6: Basis and Dimension

Problem (2). (Not from the book.) Recall that $\{E^{ij} : 1 \leq i \leq n, 1 \leq j \leq n\}$ forms a basis for $\mathrm{M}_{n\times n}(F)$, where E^{ij} is the matrix whose only nonzero entry is a 1 in the ith row and jth column. (See Example 1.6.3 or the lecture notes.) So any matrix $M \in \mathrm{M}_{n\times n}(F)$ can be written as $M = \sum_{i,j=1}^n a_{ij}E^{ij}$ for some elements $a_{ij} \in F$. Also, recall that $M^t = \sum_{i,j=1}^n a_{ij}(E^{ij})^t$ (see p. 17 and Exercise 1.3.3). Now $(E^{ij})^t = E^{ji}$, so $M^t = \sum_{i,j=1}^n a_{ij}E^{ji} = \sum_{i,j=1}^n a_{ji}E^{ij}$. If M is symmetric, then $M = M^t$, and so $0 = M - M^t = \sum_{i,j=1}^n (a_{ij} - a_{ji})E^{ij}$. Since $\{E^{ij} : 1 \leq i \leq n, 1 \leq j \leq n\}$ is linearly independent, this implies that $a_{ij} = a_{ji}$ for all i, j $(1 \leq i \leq n, 1 \leq j \leq n)$. Hence, $M = \sum_{i=1}^n a_{ii}E^{ii} + \sum_{i<j} a_{ij}(E^{ij} + E^{ji})$, i.e, M can be written as a diagonal matrix plus a linear combination of matrices of the form $E^{ij} + E^{ji}$. Therefore, the set $S = \{E^{ii} : 1 \leq i \leq n\} \cup \{E^{ij} + E^{ji} : 1 \leq i < j \leq n\}$ spans the subspace of symmetric matrices, or W. We claim that S is actually a basis for W. To show this, we need to prove that S is linearly independent. Suppose that a linear combination of the elements of this set is 0: $\sum_{i=1}^n a_{ii}E^{ii} + \sum_{i<j} a_{ij}(E^{ij} + E^{ji}) = 0$ for some elements $a_{ij} \in F$. Then $0 = \sum_{i=1}^n a_{ii}E^{ii} + \sum_{i<j} a_{ij}E^{ij} + \sum_{i<j} a_{ij}E^{ji}$. Since $\{E^{ij} : 1 \leq i \leq n, 1 \leq j \leq n\}$ is linearly independent, this implies that each $a_{ij} = 0$. Hence, S is linearly independent, and therefore a basis for W. Since S consists of $n + \frac{n^2-n}{2} = \frac{n^2+n}{2}$ elements, the dimension of W is $\frac{n^2+n}{2}$.

Problem 1.

(a) False. The empty set is a basis for the zero vector space. (See Example 1 in Sect. 1.6.)
(b) True. See Theorem 1.9.
(c) False. An infinite-dimensional vector space has no finite basis (e.g, $\mathrm{P}(F)$).
(d) False. See Examples 2 and 15 in Sect. 1.6 for two different bases for $V = \mathbb{R}^4$.
(e) True. See Corollary 1 in Sect. 1.6.

(f) False. By Example 10 in Sect. 1.6, $\mathrm{P}_n(F)$ has dimension $n+1$.
(g) False. By Example 9 in Sect. 1.6, $\mathrm{M}_{m\times n}(F)$ has dimension mn.
(h) True. See Theorem 1.10.
(i) False. For example, if we set $v_1 = (1,0)$, $v_2 = (0,1)$, and $v_3 = (1,1)$, then $\{v_1, v_2, v_3\}$ generates $\mathbb{R}^2$, but $(1,1)$ can written both as $1*v_1+1*v_2+0*v_3$ and as $0*v_1+0*v_2+1*v_3$.
(j) True. See Theorem 1.11.
(k) True. By Theorem 1.11, if a subspace of V has dimension n, then that subspace is equal to V. The only vector space that has dimension 0 is the zero vector space.
(l) True. If S is linearly independent, then S spans V, by Corollary 2 (b) (Sect. 1.6). Conversely, if S spans V, then S is linearly independent, by part (a) of that corollary.

Problem 5. No. By Example 8 (Sect. 1.6), the dimension of $\mathbb{R}^3$ is 3, and it is thus generated by a 3-element set. So, by Theorem 1.10, any linearly independent subset of $\mathbb{R}^3$ can have at most 3 elements. Therefore, $\{(1,4,-6),(1,5,8),(2,1,1),(0,1,0)\}$ cannot be linearly independent. Or, more directly, $-15(1,4,-6)-13(1,5,8)+14(2,1,1)+111(0,1,0) = 0$.

Problem 11. Since $\{u, v\}$ is a basis, V must have dimension 2. So, by Corollary 2 (b) (Sect. 1.6), to show that $\{u+v, au\}$ and $\{au, bv\}$ are bases, it is enough to show that they are linearly independent.

Suppose that $c(u+v)+d(au) = 0$ for some $c, d \in F$. Then $(c+da)u+cv = 0$, and so $c+da = 0$ and $c = 0$, by the linear independence of $\{u, v\}$. But, since $a \neq 0$, d must also be zero. So $\{u+v, au\}$ is linearly independent.

Suppose that $c(au)+d(bu) = 0$ for some $c, d \in F$. Then, $ca = 0 = db$, since $\{u, v\}$ is linearly independent. Since a and b are nonzero, $c = d = 0$. So $\{au, bv\}$ is linearly independent.

Problem 12. As in the previous problem, it is enough to show that $\{u+v+w, v+w, w\}$ is linearly independent. Suppose that $a(u+v+w)+b(v+w)+cw = 0$ for some $a, b, c \in F$. Then $au+(a+b)v+(a+b+c)w = 0$. Since $\{u, v, w\}$ is linearly independent, $a = 0$, $a+b = 0$, and $a+b+c = 0$. Solving this linear system, we see that $a = b = c = 0$, and so $\{u+v+w, v+w, w\}$ is linearly independent.

Problem 13. Subtracting the first equation from the second, we see that $x_1 = x_2$. Plugging this back into the first equation, we see that $x_2 = x_3$. Hence, the solutions to this system are precisely triplets (x_1, x_2, x_3) of the form $(x, x, x) = x(1,1,1)$. So $\{(1,1,1)\}$ spans the subspace of $\mathbb{R}^3$ consisting of solutions to the given system. Also, the set $\{(1,1,1)\}$ is clearly linearly independent. Thus, $\{(1,1,1)\}$ is a basis for the subspace in question.

Problem 29.

(a) Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for $W_1 \cap W_2$. Since $\{u_1, u_2, \ldots, u_k\}$ is a linearly independent set that is contained in W_1, it can be extended to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$ for W_1, by Corollary 2 (c) (Sect. 1.6). Similarly, $\{u_1, u_2, \ldots, u_k\}$ can be extended to a basis $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_p\}$ for W_2. Now, each element of W_1 can be written as a linear combination of $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$, and each element of W_2 can be written as a linear combination of $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_p\}$. So each element of $W_1 + W_2$ can be written as a linear combination of $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_p\}$, by definition of sum. In other words, $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_p\}$ spans $W_1 + W_2$.

Also, note that $\operatorname{span}(u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m) \cap \operatorname{span}(w_1, w_2, \ldots, w_p) = \{0\}$, since if there is a nonzero element t that is in both sets, then it is contained in $W_1 \cap W_2$, and hence must be a linear combination of $\{u_1, u_2, \ldots, u_k\}$. Writing $t = a_1u_1 + \cdots + a_ku_k$ and $t = b_1w_1 + \cdots + b_pw_p$ for some $a_1, \ldots a_k, b_1, \ldots, b_p \in F$, we get $a_1u_1 + \cdots + a_ku_k - b_1w_1 - \cdots - b_pw_p = 0$. Since we assumed that t is nonzero, this gives a nontrivial linear combination of $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_p\}$, contradicting the assumption that this is a linearly independent set. Hence, there can be no nonzero element in $\operatorname{span}(u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m) \cap \operatorname{span}(w_1, w_2, \ldots, w_p)$.

Now, suppose that $a_1u_1 + a_2u_2 + \cdots + a_ku_k + b_1v_1 + b_2v_2 + \cdots + b_mv_m + c_1w_1 + c_2w_2 + \cdots + c_pw_p = 0$ for some $a_1, \ldots a_k, b_1, \ldots, b_m, c_1, \ldots, c_p \in F$. Then $a_1u_1 + a_2u_2 + \cdots + a_ku_k + b_1v_1 + b_2v_2 + \cdots + b_mv_m = -c_1w_1 - c_2w_2 - \cdots - c_pw_p$, and so each side of this expression must equal zero, by the previous paragraph. Since $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_p\}$ are linearly independent sets, this means that $a_1 = \cdots = a_k = b_1 = \cdots = b_m = c_1 = \cdots = c_p = 0$. Hence, $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_p\}$ is linearly independent and thus is a basis for $W_1 + W_2$. In particular, $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = k + m + p$.

Now, looking at our bases for $W_1 \cap W_2$, W_1, and W_2 we see that $\dim(W_1 \cap W_2) = k$, $\dim(W_1) = k + m$, and $\dim(W_2) = k + p$. So $\dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = k + m + p$, and hence $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

(b) Since $V = W_1 + W_2$, by part (a), $\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. Then $\dim(V) = \dim(W_1) + \dim(W_2) \Leftrightarrow \dim(W_1 \cap W_2) = 0 \Leftrightarrow W_1 \cap W_2 = \{0\} \Leftrightarrow V$ is the direct sum of W_1 and W_2.

Problem 31.

(a) $W_1 \cap W_2 \subseteq W_2$, so $\dim(W_1 \cap W_2) \leq \dim(W_2) = n$, by Theorem 1.11.

(b) By Problem 29, $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = m + n - \dim(W_1 \cap W_2) \leq m + n$, since $\dim(W_1 \cap W_2)$ is a nonnegative integer.