MATH 110: LINEAR ALGEBRA
PRACTICE MIDTERM #2

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*Note

The theorems in sections 5.1 and 5.2 each have two versions, one stated in terms of linear operators, one in terms of matrices. The book states most of them in terms of linear operators, whilst in the lecture notes, they are mostly stated in terms of matrices. For example, compare Theorem 5.5 and its corollary in the book with Theorem 5 and its corollary in the lecture notes; also compare Theorem 5.6 in the book with the computation of the characteristic polynomial of a diagonalizable matrix done in lectures. In each case, one can derive one version from the other, by considering L_A and $[T]_\beta$. In these solutions I'll reference theorems in the book, but often I literally mean the matrix version of that theorem.

Problem 1. Let $\gamma = \{e_1, \ldots, e_k\}$ be an ordered basis for W. As W is a subspace of V, we may extend γ to an ordered basis $\beta = \{e_1, \ldots, e_n\}$ for V. Note that since $W \neq V$, $\dim(W) < \dim(V)$, so $k < n$. Let $\beta^* = \{f_1, \ldots, f_n\}$ be the dual basis to β. So $f_i \in V^*$. By definition of dual basis,

$$f_{k+1}(e_i) = \delta_{k+1,i}.$$

But then $f_{k+1}(u) = 0$ for each $u \in \gamma$. As γ is a basis for W, $f_{k+1}(u) = 0$ for each $u \in \mathrm{span}(\gamma) = W$ (by linearity of f_{k+1}). But $f_{k+1}(e_{k+1}) = 1$, so $f_{k+1} \neq 0$. Thus f_{k+1} is as desired.

Alternatively, one could define f using the method f_{k+1} is defined. Let γ and β be as above. Using Theorem 2.6, there is a unique linear $f : V \to F$ (where V is over the field F) satisfying $f(e_i) = 0$ for each $i \leq k$ and $i > k$, and $f(e_{k+1}) = 1$.

Problem 2. Recall that to find an LU-decomposition for a matrix, we perform Gaussian elimination, hoping that we'll never have to swap any rows or columns in order to get our next pivot point. If we do have to, the decomposition is more complicated. The proof for this problem is motivated by the following observation: if A is invertible, one can reduce A into a lower triangular unit matrix by performing a series of these operations:

(1) Row swaps;
(2) Multiplying a column by a non-zero scalar;
(3) Adding a multiple of column i to column j, where $i < j$.

If you were unable to do this problem, I suggest you stop reading here and try to first prove the above statement, and then use this fact to prove the appropriate LU-decomposition can be found.

Here is a sketch of how to obtain the appropriate LU-decomposition from the above process (it's not a complete proof). Note that (1) is done by multiplying on the left with a permutation matrix, and (2) and (3) are each done by multiplying on the right by an invertible

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upper triangular matrix. Thus we end up with

$$L = P_1 P_2 \dots P_k A U_1 U_2 \dots U_j.$$

Using some previous homework problems, this means $L = P'AU'$, which leads to $A = P'^{-1}LU^{-1}$, which is as required.

The above process also motivates a slicker, though less intuitive proof. This works in the same way as the proof of the existence of LU-decomposition done in lectures. We use induction on the size of A to prove: if A is an $n*n$ invertible matrix, there are a permutation matrix P, lower triangular unit L and upper triangular invertible U such that $A = PLU$.
If A is $1*1$, $P = L = [1]$ and $U = A$.
Suppose $n > 1$. As A is invertible, its first column is non-zero, so if $A_{11} = 0$ we may swap the first row with another with some permutation matrix P', so that $(P'A)_{11} \neq 0$. We may then perform operations (2) and (3) above to produce a matrix A' such that $A'_{11} = 1$ and $A'_{1i} = 0$ for $i > 1$. There is an invertible upper triangular matrix U' which does this when multiplying on the right. So we get

$$A' = P'AU' = \begin{bmatrix} 1 & 0 \\ X & S \end{bmatrix}$$

where A' is written as a block matrix with a $1*1$ upper-left block. Now A' is invertible as it is written as the product of invertible matrices. This means S is invertible (if $Sv = 0$, then setting $v' = [0; v^t]^t$, $A'v = 0$, so $v' = 0$, so $v = 0$). So we can apply the inductive hypothesis and get $S = P_1 L_1 U_1$. Substituting this and factoring the above block matrix, we get

$$P'AU' = \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P_1^{-1}X & L_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_1 \end{bmatrix}$$

(Factor it in two steps to obtain this.) Call the three matrices in the above product P'', L, U''. Note that these are a permutation, lower triangular unit, and upper triangular invertible, respectively, as P_1, L_1 and U_1 were. U' is invertible, (as is P'), so

$$A = P'^{-1}P''LU''U'^{-1} = PLU,$$

where $P = P'^{-1}P''$ (so is a permutation, by homework problems) and $U = U''U'^{-1}$ (so is upper triangular and invertible, by homework problems). Thus we have the required decomposition.

Problem 3. Let L_i and U_i be the upper-left $i*i$ blocks of L and U respectively. Then L_i is lower triangular unit and U_i is upper triangular. Partition L as a $2*2$ block matrix with L_i the upper-left block. Then we have

$$A = LU = \begin{bmatrix} L_i & 0 \\ W_i & X_i \end{bmatrix} \begin{bmatrix} U_i & Y_i \\ 0 & Z_i \end{bmatrix}.$$

(W_i is the lower-left $(n-i)*i$ block of L, etc.) Block multiplying, we have $A_i = L_iU_i + 0.0 = L_iU_i$. So with $i = 1$, we have $A_{11} = L_{11}U_{11} = 1.U_{11}$. For any i,

$$\det(A_i) = \det(L_iU_i) = \det(L_i)\det(U_i) = (\Pi_{j=1}^{j=i}1)(\Pi_{j=1}^{j=i}U_{i,jj}) = \Pi_{j=1}^{j=i}U_{i,jj}.$$

(Here I've used the fact that det preserves products, and the det of a triangular matrix is the product of its diagonal elements, and L_i is unit.) As U is invertible triangular, its diagonal elements are non-zero, so the determinants here are non-zero. Therefore for $i > 1$,

$$\det(A_i)/\det(A_{i-1}) = (\Pi_{j=1}^{j=i} U_{i,jj})/(\Pi_{j=1}^{j=i-1} U_{i,jj}) = U_{i,ii} = U_{ii}.$$

Problem 4. False. The characteristic polynomial of a matrix M over $\mathbb{R}$ may have no factors over $\mathbb{R}$, but split into factors each with multiplicity 1 over $\mathbb{C}$. In this situation M would have no eigenvalues over $\mathbb{R}$, and so be non-diagonalizable over $\mathbb{R}$, by (the matrix version of) Theorem 5.6. However, it would be diagonalizable over $\mathbb{C}$, by the corollary to Theorem 5.5. The canonical example is a transformation which rotates the $\mathbb{R}^2$ plane by 90 degrees. Clearly this linear transformation has no eigenvectors in $\mathbb{R}^2$, which (essentially by definition) is equivalent to having no eigenvalues, which is equivalent to its characteristic polynomial having no factors (by theorem 5.2). A matrix representing such a transformation is

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

($-M$ also rotates 90 degrees, in the opposite direction.) M is certainly over $\mathbb{R}$. The characteristic polynomial of M is $p(x) = x^2 + 1$. It has no factors over $\mathbb{R}$, but over $\mathbb{C}$, $p(x) = (x - i)(x + i)$, so p is as in the above discussion, so M is diagonalizable over $\mathbb{C}$.

Problem 5. True. Suppose $Ax = 0$ has exactly one solution. Clearly this solution is $x = 0$. Then nullity$(A) = 0$, so A is invertible, so $Ax = b$ iff $x = A^{-1}b$, so $A^{-1}b$ is the unique solution.
Conversely, suppose $Ax = 0$ has multiple solutions (there can't be no solutions as $x = 0$ solves it). So $\mathbf{N}(A)$ has more than 1 element. If the equation $Ax = b$ has no solutions in x, then there is certainly not a unique solution, so we're done. So suppose there is a solution, and that $Ax_1 = b$. Then given any y, we have

$$Ay = b \iff Ay - Ax_1 = 0 \iff A(y - x_1) = 0 \iff y - x_1 \in \mathbf{N}(A).$$

Thus the complete solution set to $Ax = b$ is

$$\{x_1 + x | x \in \mathbf{N}(A)\}.$$

As $\mathbf{N}(A)$ has more than 1 element, so does the solution set above (if $x_1 + x = x_1 + y$ then $x = y$). Therefore there is not a unique solution to $Ax = b$.

Problem 6. True. The eigenvalues of such a matrix are the diagonal entries (problem 9 of 5.1). As these are distinct, (the matrix version of) the corollary to Theorem 5.5 shows that A is diagonalizable (or the corollary to Theorem 5 in the lecture notes is direct).

Problem 7. True. For given $\lambda \in F$, λ is an eigenvalue iff $\det(A - \lambda I) = 0$. But we have

$$0 = \det(0) = \det((A - 2I)(A - 3I)(A - \pi I)) = \det(A - 2I)\det(A - 3I)\det(a - \pi I),$$

as det preserves products. Therefore at least one of the determinants in the product is 0, so at least one of 2, 3 and π is an eigenvalue.

Problem 8. Let C_i be the $i * i$ square matrix in the upper-right corner of C. Define the series a_n by $a_1 = 1$, and for $n > 1$, $a_n = a_{n-1}$ if n is odd, and $a_n = -a_{n-1}$ if n is even. So the series begins $1, -1, -1, 1$, and repeats this pattern every four terms.
Claim:

$$\det(C_n) = a_n \Pi_{i=1}^{i=n} c_i.$$

Proof: Clearly $\det(C_1) = c_1 = a_1c_1$.
Suppose $n > 1$. Expanding along the first column,

$$\det(C_n) = (-1)^{n+1}c_n \det(\tilde{C}_{n1}) = (-1)^{n+1}c_n \det(C_{n-1}),$$

as it is clear that $\tilde{C}_{n1} = C_{n-1}$. By induction then,

$$\det(C_n) = (-1)^{n+1}c_n a_{n-1}\Pi_{i=1}^{i=n-1}c_i = (-1)^{n+1}a_{n-1}\Pi_{i=1}^{i=n}c_i.$$

If n is odd, $(-1)^{n+1} = 1$, so $(-1)^{n+1}a_{n-1} = a_{n-1} = a_n$. If n is even $(-1)^{n+1} = -1$ and $a_n = -a_{n-1}$, so it works.

Problem 9. Let p be the characteristic polynomial of A,

$$p(x) = (-1)^n x^n + c_{n-1}x^{n-1} + \ldots + +c_1x + c_0.$$

As A is diagonalizable, by the Cayley-Hamilton theorem for diagonalizable matrices,

$$p(A) = (-1)^n A^n + c_{n-1}A^{n-1} + \ldots + c_1A + c_0I = 0.$$

Now, as A is invertible, we get

$$-c_0A^{-1} = (-1)^n A^{n-1} + c_{n-1}A^{n-2} + \ldots + c_1I.$$

Now as long as $c_0 \neq 0$, we can divide by $-c_0$ to get the sort of expression we need (note the c_i's here are different to those in the question). But as p is the characteristic polynomial of A, so by exercise 20 of section 5.1, $c_0 = \det(A)$, and $\det(A) \neq 0$ as A is invertible.

Problem 10. We have $\beta = \{1, x, x^2, x^3\}$. Recall that β^* is the ordered basis $\{f_0, f_1, f_2, f_3\}$ for $\mathrm{P}_3(\mathbb{R})^*$, where

$$f_i(x^j) = \delta_{ij},$$

or equivalently, the f_i are the linear functionals which "project on β co-ordinates", so letting $q \in \mathrm{P}_3(\mathbb{R})$, if $q = c_3x^3 + c_2x^2 + c_1x^1 + c_0$, then as c_0 is q's coefficient of 1, $f_0(q) = c_0$, and likewise $f_1(q) = c_1$, etc.
Now, recall that the columns of $[T]_\beta^{\beta^*}$ are given by expressing the elements of $T(\beta)$ in β^* co-ordintates. That is, column $i+1$ is $[T(x^i)]_{\beta^*}$.
For example, let's calculate column 2. So this is $[T(x)]_{\beta^*}$. So we need to know what $T(x)$ does, and express it in terms of the projection functionals mentioned above.
Now $T : \mathrm{P}_3(\mathbb{R}) \to \mathrm{P}_3(\mathbb{R})^*$, so $T(x)$ is a linear functional on $\mathrm{P}_3(\mathbb{R})$; that is, $T(x) : \mathrm{P}_3(\mathbb{R}) \to \mathbb{R}$. So we need to look at what $T(x)$ does given input some $q \in \mathrm{P}_3(\mathbb{R})$. By definition,

$$T(x)(q) = \int_0^1 xq(x)\ dx.$$

Expressing q in the β basis, $q = c_3x^3 + c_2x^2 + c_1x^1 + c_0$, say, then

$$\begin{aligned} T(x)(q) &= \int_0^1 x(c_3x^3 + c_2x^2 + c_1x^1 + c_0)\ dx \\ &= (1/5)c_3 + (1/4)c_2 + (1/3)c_1 + (1/2)c_0. \end{aligned}$$

But note that this

$$= (1/5)f_3(q) + (1/4)f_2(q) + (1/3)f_1(q) + (1/2)f_0(q).$$

As q was arbitrary, we have

$$T(x) = (1/5)f_3 + (1/4)f_2 + (1/3)f_1 + (1/2)f_0.$$

This expresses $T(x)$ in the form required, and so the second column of $[T]_\beta^{\beta^*}$ is $[\frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}\ \frac{1}{5}]^t$. (Note I had to reverse the order from that in the calculation because of the set ordering on β and β^*).
The other columns are computed in exactly the same fashion.

Problem 11. Note that the T given in this problem has domain V, and one is supposed to show $T : V' \to W'$. Strictly speaking, this doesn't make sense unless $V = V'$, as a function only has one domain. So the problem is really to find a linear T' and a subspace V' of V so that $T' : V' \to W'$ is an isomorphism, and T' agrees with T on V'. With these requirements, there's only one possible choice. We need V' to be the set of vectors in V that T maps into W'. So let

$$V' = \{v \in V | T(v) \in W'\}.$$

Then V' is a subspace: suppose $v_1, v_2 \in V'$. Then $T(v_1 + v_2) = T(v_1) + T(v_2) \in W'$, as W' is cosed under $+$. Similarly, V' is closed under multiplication and $0 \in V'$.
Define T' to be the restriction of T to V' (i.e. T'(v)=T(v) for each $v \in V'$). Then clearly $T' : V' \to W'$ and T' is linear because T is.
T' is also 1-1 because T is. The key point is $\mathbf{Rg}(T') = W'$. This is because $\mathbf{Rg}(T) = W$, so given $w \in W'$, there is $v \in V$ such that $T(v) = w$, and by definition of V', $v \in V'$, and $T'(v) = T(v) = w$. Thus $w \in \mathbf{Rg}(T')$, so T' is onto.
Therefore T' is an isomorhpism.

Problem 12. We use Gaussian elimination, recording the matrices needed to perform matrix operations, to find an LU-decomposition. Perform the following 3 operations:

(1) Swap rows 1 & 2, by left-multiplying with the permutation matrix P.
(2) Add $-5*$(Row 1) to (Row 3), by left-multiplying with L_1.
(3) Add $2*$(Row 2) to (Row 3), by left-multiplying with L_2.

Call the resulting (upper triangular) matrix U. The matrices involved are

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix};\ L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix};\ U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix}.$$

Combining the above process, we have $L_2 L_1 P A = U$. Each L_i and P are invertible, so $A = P^{-1} L_1^{-1} L_2^{-1} U = PLU$, setting

$$L = L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix};$$

U has no zero rows, so we don't need to alter the dimensions of the matrices, so we have an LU-decomposition of A.
Now $\det(A) = \det(PLU) = \det(P)\det(L)\det(U)$. P is obtained by swapping two rows of I, so $\det(P) = -1$. L and U are triangular, so their determinants are the product of their eigenvalues. Thus $\det(L) = 1$ and $\det(U) = -4$. So $\det(A) = 4$.
Now we solve the equation $Ax = [1; -1; 0]^t = b$ (in this case we already know there is exactly one solution as A is invertible). First we look for solutions to $PLy = b$, or equivalently,

$$Ly = P^t b = [-1; 1; 0]^t.$$

Solving by substitution, we get $y = [-1; 1; 7]^t$. Now we solve

$$Ux = [-1; 1; 7]^t.$$

Solving by substitution, we get $x = [-2; 23/4; -7/2]^t$.

Problem 13. Computing the characteristic polynomial $p(x) = \det(A - xI)$, we get $p(x) = (7 - x)(6 - x)(1 - x)$. Thus A has 3 distinct eigenvalues, and as A is $3 * 3$, by the corollary to theorem 5.5 (matrix version), A is diagonalizable.