Solutions to Practice Midterm 1

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Author's note: I've attempted to be as complete as possible in my solution, hence the wordiness. In the midterm, you can probably miss one or two small steps.

(Question 1): True or False (with justification): If A and B are n-by-n matrices with entries from F, then $AB = 0$ if and only $BA = 0$.

Solution. False. Counterexample: if $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then $AB = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ but $BA = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. $\square$

(Question 2): True or False (with justification): If A and B are n-by-n matrices with entries from $\mathbb{R}$, then $AB = 7I_n$ if and only $BA = 7I_n$.

Solution. True. First, a general statement. We claim that a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism iff it is one-to-one. Now, $\Rightarrow$ is obvious. For the converse, suppose T is one-to-one. Then $n = \dim(\mathbb{R}^n) = \dim N(T) + \dim R(T)$. Since T is one-to-one, $\dim N(T) = 0$, and so $\dim R(T) = n$. Since $R(T) \subseteq \mathbb{R}^n$, this implies $R(T) = \mathbb{R}^n$.

Now look at L_A and L_B as linear maps $\mathbb{R}^n \to \mathbb{R}^n$. By symmetry, it suffices to prove $\implies$. Since $L_A L_B = 7 \cdot 1_V$ is one-to-one, L_A is also one-to-one. By the above paragraph, L_A is an isomorphism and thus has an inverse U, i.e. $UL_A = L_A U = 1_V$. So,

$$UL_A L_B = L_B \implies 7U = L_B \implies U = \frac{1}{7} L_B.$$

Thus, $\frac{1}{7} L_B$ is the inverse of L_A and $\frac{1}{7} L_B L_A = 1_V \implies L_B L_A = 7 \cdot 1_V$. $\square$

(Question 3): True or False (with justification): If $x, y \in V$ and $a, b \in F$, then $ax + by = 0$ if and only if x is a scalar multiple of y, or y is a scalar multiple of x.

Proof. False. This is a trick question: we can take $a = b = 0$ and then x and y can be any vectors, say $(1, 0)$ and $(0, 1)$ in $V = \mathbb{R}^2$. $\square$

(Question 4): True or False (with justification): For $A \in M_{m \times n}(F)$, $[L_A]_\beta^\gamma = A$ if β and γ are the standard bases of F^n and F^m.

Proof. True, essentially by the definition of L_A. $\square$

(Question 5): Suppose that $T : V \to V$ is a linear transformation. Prove that $T^2 = 0$ if and only if the range of T is a subspace of the null space of T.

Proof. Suppose $T^2 = 0$. We wish to prove $R(T) \subseteq N(T)$. To do that, let $v \in R(T)$. Then $v = T(v')$ for some $v' \in V$. But now since $T(v) = T^2(v') = 0$, we have $v \in N(T)$. This proves $R(T) \subseteq N(T)$.

Conversely, suppose $R(T) \subseteq N(T)$. To show $T^2 = 0$, we have to show $T(T(v)) = 0$ for any $v \in V$. But note that $T(v) \in R(T)$, and since $R(T) \subseteq N(T)$, we have $T(v) \in N(T)$. Hence $T(T(v)) = 0$ as we hoped to prove. $\square$

(Question 6): Let $V = P(\mathbb{R})$ be the vector space of polynomials with real coefficients. For each $i \geq 0$, let $f_i \in \mathcal{L}(V, \mathbb{R})$ be the linear transformation that maps a polynomial $p(x)$ to the value $p^{(i)}(0)$ of its i-th derivative at 0. (The 0-th derivative of p is p itself.) Show that the linear transformations $f_0, f_1, f_2, ...$ are linearly independent vectors in the vectors space $\mathcal{L}(V, \mathbb{R})$.

Proof. Suppose not, i.e. assume the f_i's are linearly dependent. Then we can find scalars $a_0, a_1, \ldots, a_n \in \mathbb{R}$, not all zero, such that

$$a_0 f_0 + a_1 f_1 + \cdots + a_n f_n = 0.$$

Dropping terms on the right which are zero, we may assume $a_n \neq 0$. Let the above linear functional act on $p(x) = x^n$. Then we have:

$$a_0 f_0(x^n) + a_1 f_1(x^n) + \cdots + a_n f_n(x^n) = 0.$$

However, $\frac{d^i}{dx^i}(x^n) = n(n-1)\ldots(n-i+1)x^{n-i}$. So $f_0(x^n) = f_1(x^n) = \cdots = f_{n-1}(x^n) = 0$, but $f_n(x^n) = n!$. The above equation then gives $a_n n! = 0 \implies a_n = 0$ which contradicts our choice of a_n. $\square$

(Question 7): In the notation of Question 6, show that the linear transformation $g \in \mathcal{L}(V, \mathbb{R})$ where $g(p) = p(1)$ is not in the span of the f_i.

Proof. Suppose g is in the span of f_i's. Then g can be expressed as a *finite* linear combination of the f_i's, i.e.

$$g = a_0 f_0 + a_1 f_1 + \cdots + a_n f_n, \quad \text{for some scalars } a_0, \ldots, a_n \in \mathbb{R}.$$

Now put in $p(x) = x^{n+1}$ on both sides. As we have proven in question 6, $f_0(x^{n+1}) = f_1(x^{n+1}) = \cdots = f_n(x^{n+1}) = 0$. This gives 1 on the left hand side, and 0 on the right hand side - an absurd conclusion. $\square$

(Question 8): Let $T : V \to W$ be a linear transformation. Suppose that $x_1, ..., x_r$ are linearly independent elements of $N(T)$ and that $v_1, ..., v_s$ are vectors in V such that $T(v_1), ..., T(v_s)$ are linearly independent. Show that the $r + s$ vectors $x_1, ..., x_r, v_1, ..., v_s$ are linearly independent.

Proof. The proof is rather standard: suppose we have scalars $a_1, \ldots, a_r, b_1, \ldots, b_s$ such that

$$a_1 x_1 + \cdots + a_r x_r + b_1 v_1 + \cdots + b_s v_s = 0.$$

Applying T to both sides, and noting that each $x_i \in N(T)$, we have $b_1 T(v_1) + \cdots + b_s T(v_s) = 0$. Since the $T(v_i)$'s are linearly independent, all b_i's must be zero. Substituting $b_i = 0$, we get

$$a_1 x_1 + \cdots + a_r x_r = 0.$$

Since the x_i's are linearly independent, all the a_i's must also be zero. Thus $x_1, \ldots, x_r, v_1, \ldots, v_s$ are linearly independent. $\square$

(Question 9): Construct a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(T(T(v))) = 0$ for all $v \in V$ but $T(T(v))$ is nonzero for some v. Do not forget to show that T is a linear transformation.

Proof. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Let T be the linear map defined by:

$$T(e_1) = e_2, \quad T(e_2) = e_3, \quad T(e_3) = 0.$$

Explicitly, we must have $T(a, b, c) = T(ae_1 + be_2 + ce_3) = ae_2 + be_3 = (0, a, b)$. Thus T corresponds to the matrix $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. An easy computation shows that $T^2 \neq 0$ but $T^3 = 0$. $\square$

(Question 10): Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_m)$ be given nonzero tuples with rational entries, and define the matrix $A \in M_{m\times n}(\mathcal{Q})$ by $A_{ij} = y_i \cdot x_j$. Let $L_A : \mathcal{Q}^n \to \mathcal{Q}^m$ be the linear transformation defined by multiplying a vector in $\mathcal{Q}^n$ by the matrix A. Compute the rank and nullity of L_A. Justify your answers.

Proof. Write x and y as column vectors. Then

$$A = \begin{pmatrix} y_1x_1 & y_1x_2 & \dots & y_1x_n \\ y_2x_1 & y_2x_2 & \dots & y_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_mx_1 & y_mx_2 & \dots & y_mx_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} = yx^t.$$

Writing vectors in $\mathcal{Q}^n$ as column vectors, we see that L_A is simply left multiplication by the matrix $A = yx^t$, i.e. $v \mapsto (yx^t)v = y(x^tv)$. Since x^tv is a scalar, $y(x^tv)$ must be a multiple of y. Hence $R(L_A) \subseteq \operatorname{span}(\{y\})$. On the other hand, since $x \neq 0$, the map $v \mapsto x^tv$ is not the zero map. Hence, $R(L_A) = \operatorname{span}(\{y\})$, which is of dimension 1 because $y \neq 0$. Having known the rank, the nullity is a direct consequence of the dimension theorem: $\dim N(T) = n - 1$.

$\square$