Math 110 *Practice* Final Exam

You are allowed to use 1 sheet (8 1/2 by 11 inches, both sides) of notes. Otherwise this is a closed book, closed notes, closed calculator, closed computer, closed PDA, closed cellphone, closed network, open brain exam.

You get one point each for filling in the 4 lines below All other questions are worth 10 points.

Fill in the questions below. **Then stop and wait until we tell you to turn the page and start the rest of the exam. Do not start reading the rest of the exam until we tell you to start.**

After you start, read all the questions on the exam before you answer any of them, so you do the ones you find easier first.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

(1) (1 point) What is your name?

(2) (1 point) What is your TA's name?

(3) (1 point) What is the name of the person sitting to your left? ("aisle" is a possible answer.)

(4) (1 point) What is the name of the person sitting to your right? ("aisle" is a possible answer.)

This practice exam has more questions than we will ask in the actual exam.

This practice final exam mostly covers material after the second midterm. The actual final exam will cover the whole semester, with somewhat more emphasis on material after the second midterm. Earlier practice midterms remain good sources for study for material earlier in the semester. See also on-line exams from earlier semesters from other instuctors.

(Question 1): Let T be an upper triangular n by n matix with entries from $\mathbb{C}$., with all distinct diagonal entries. In this question we will show how to compute $F = f(T)$ for any polynomial $f(.)$.

1.1. (3 points) Show that $F_{ii} = f(T_{ii})$.

1.2. (3 points) Show that $F \cdot T = T \cdot F$.

1.3. (3 points) By equating $(F \cdot T)_{i,i+1} = (T \cdot F)_{i,i+1}$, write down a formula for $F_{i,i+1}$.

1.4. (7 points) Generalize the last part, giving an explicit formula for $F_{i,i+k}$, the entries on the k-th superdiagonal of F, in terms of values of F on superdiagonals 0 through $k-1$. Why did we assume T had all distinct eigenvalues?

1.5. (4 points) Explicitly compute $\cos\left(\begin{bmatrix} \frac{\pi}{4} & 7 \\ 0 & -\frac{\pi}{4} \end{bmatrix}\right)$.

(Question 2): Let A be m-by-n and B be n-by-m, so that the two products $A \cdot B$ and $B \cdot A$ exist. In this question we will show that the eigenvalues of $A \cdot B$ and $B \cdot A$ are closely related.

2.1. Suppose $m = n$ and either A or B is nonsingular. Show that $A \cdot B$ and $B \cdot A$ have the same eigenvalues. Hint: Show $A \cdot B$ and $B \cdot A$ are similar.

2.2. (Harder) Again suppose $m = n$, but A and B are otherwise arbitrary (so they may be singular). Show that $A \cdot B$ and $B \cdot A$ still have the same eigenvalues. Hint: When $\lambda = 0$, show that $\det(\lambda I - A \cdot B) = \det(\lambda I - B \cdot A)$. When $\lambda \neq 0$, show that

$$\begin{aligned} \det(\lambda I_n - A \cdot B) &= \det\left(\begin{bmatrix} I_n & B \\ A & \lambda I_n \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} \lambda I_n & 0 \\ 0 & I_n \end{bmatrix} \cdot \begin{bmatrix} I_n & B \\ A & \lambda I_n \end{bmatrix} \cdot \begin{bmatrix} I_n & 0 \\ 0 & \lambda^{-1} I_n \end{bmatrix}\right) \\ &= \det(\lambda I_n - B \cdot A) \end{aligned}$$

by evaluating the $2n$-by-$2n$ determinants in 2 different ways. This is a varation on the proof in the notes that $\det(A \cdot B) = \det(B \cdot A)$. Conclude that the characteristic polynomials of $A \cdot B$ and $B \cdot A$ are the same.

2.3. (Still harder) Now suppose $m > n$. Show that the m eigenvalues of $A \cdot B$ are the same as the n eigenvalues of $B \cdot A$, including multiplicities, along with an additionl $m - n$ eigenvalues equal to 0. Hint: When $\lambda \neq 0$, show that

$$\det(\lambda I_m - A \cdot B) = \det\begin{bmatrix} I_n & B \\ A & \lambda I_m \end{bmatrix} = \lambda^{m-n} \det(\lambda I_n - B \cdot A)$$

by a variation on the above hint.

(Question 3): Let $< \cdot, \cdot >$ be the standard dot product on $\mathbb{R}^n$. Let $A = \begin{bmatrix} 1 & 5 & 8 \\ 1 & -1 & 4 \\ 1 & 5 & 2 \\ 1 & -1 & -2 \end{bmatrix}$.

3.1. With respect to this inner product, compute the QR decomposition of A.

3.2. Let $y = [36, 72, -108, -36]^t$. Find the x minimizing $\|A \cdot x - y\|$.

(Question 4): Let P be an n-by-n permutation matrix with $P_{i,n+1-i} = 1$, i.e. P is the identity I flipped left to right.

4.1. How is $P \cdot X \cdot P$ is related to X?

4.2. Show that A and A^t have the same Jordan Form, for any A, and hence are similar. Hint: Reduce to the case of a single Jordan block and use part 2.

(Question 5): Let A be square and invertible, with $A = QR$ being the QR decomposition. Show how to express an LU decomposition of $A^* \cdot A$ in terms of Q and R.

(Question 6): Let A be a 5 by 5 matrix with entries from field F, suppose $\{u_1, u_2\}$ are two eigenvectors with common eigenavalue c, and $\{v_1, v_2, v_3\}$ are three eigenvectors with common eigenvalue $d \neq c$, and $\{u_1, u_2, v_1, v_2, v_3\}$ is an ordered basis of F^5. Let $E_\lambda = NullSpace(A - \lambda_I)$. Show that $E_c = span(u_1, u_2)$, $E_d = span(v_1, v_2, v_3)$, and that c and d are the only eigenvalues of A.

(Question 7): Let $A \in M_{n\times n}(\mathbb{C})$ and let $\{u_1, ..., u_r, v_{r+1}, ..., v_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Suppose $A \cdot u_i = c \cdot u_i$ and $A \cdot v_j = d \cdot v_j$ with $c \neq d$.

7.1. Show that $A^* \cdot u_i = \bar{c} \cdot u_i$.

7.2. Show that the space V spanned by the columns of A is the same as the space spanned by the columns of A^*, and that V must either be E_c, E_d or $\mathbb{C}^n$.

(Question 8): (True/False, with justification)

8.1. There exists a unitary Q such that $Q^* \cdot A \cdot Q$ is diagonal, where A is 10 by 10 and $A_{ij} = (i + j) \cdot \sin(i - j)$

8.2. There exists a unitary Q such that $Q^* \cdot A \cdot Q$ is diagonal, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

8.3. If J and K are in Jordan Canonical Form, then $J \cdot K = K \cdot J$.

8.4. If J and K are both diagonalizable and have the same eigenvectors, then $J \cdot K = K \cdot J$.

8.5. If A is 2 by 2 and real, then there is a real invertible Q and upper triangular T such that $A = Q^{-1} \cdot T \cdot Q$.

8.6. Let $x = [1, 2, 3, -4, 5]^t$ and $y = [5, -4, 3, 3, 1]^t$. There is a probability matrix M such that $M \cdot x = y$.

(Question 9): Let $f(x)$ be a polynomial of degree n. Prove that all n by n complex matrices with characteristic polynomial $f(x)$ are similar if and only if all of f's roots are distinct.

(Question 10): Let $V = \mathbb{C}^n$ and let A be an n-by-n matrix over $\mathbb{C}$, Suppose that there is a n-by-n matrix B such that $< x, A \cdot y >=< B \cdot x, y >$ for all $x, y \in V$, where $< ., . >$ is the standard dot product. Show that $B = A^*$.

(Question 11): Let V be an inner product space and $x \in V$. show that the set of vectors in V that are orthogonal to x forma a subspace of V.