Math 110 Final Exam with Answers

NAME (1 pt): _____

TA (1 pt): _____

Name of Neighbor to your left (1 pt): _____

Name of Neighbor to your right (1 pt): _____

Instructions: You are allowed to use 1 sheet (8 1/2 by 11 inches, both sides) of notes. Otherwise this is a closed book, closed notes, closed calculator, closed computer, closed PDA, closed cellphone, closed mp3 player, closed network, open brain exam.

You get one point each for filling in the 4 lines at the top of this page. All other questions are worth 10 points.

Fill in the questions at the top of the page. **Then stop and wait until we tell you to turn the page and start the rest of the exam. Do not start reading the rest of the exam until we tell you to start.**

After you start, read all the questions on the exam before you answer any of them, so you do the ones you find easier first.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

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Question 1. (10 points) (version 1) Let $P(\mathbb{R})$ be the vector space of all real polynomials. Define the linear map $T : P(\mathbb{R}) \to P(\mathbb{R})$ by $T(f) = f''$, the second derivative.

Part 1: Show that T is onto but not one-to-one.

Answer: If $g(x) = \sum_{i=0}^{d} g_i x^i$, *then* $T(f) = g$ *where* $f(x) = f_0 + f_1 x + \sum_{i=0}^{d} \frac{g_i x^{i+2}}{(i+1)(i+2)}$ *(so* T *is onto) and* f_0 *and* f_1 *are arbitrary (so* T *is not one-to-one).*

Part 2: Describe all eigenvectors of T, i.e. nonzero polynomials such that $f''(x) = \lambda f(x)$ for some λ.

Answer: Since the degree of f'' *is less than the degree of* f, f'' *cannot be a nonzero multiple of* f. *Therefore* $\lambda = 0$ *and* $f''(x) = 0$ *is satisfied by all linear polynomials* $f(x) = f_0 + f_1 x$.

Question 1. (10 points) (version 2) Let $P(\mathbb{C})$ be the vector space of all complex polynomials. Define the linear map $S : P(\mathbb{C}) \to P(\mathbb{C})$ by $S(g) = g''$, the second derivative.

Part 1: Show that S is onto but not one-to-one.

Answer: If $f(x) = \sum_{i=0}^{d} f_i x^i$, *then* $S(g) = f$ *where* $g(x) = g_0 + g_1 x + \sum_{i=0}^{d} \frac{f_i x^{i+2}}{(i+1)(i+2)}$ *(so* S *is onto) and* g_0 *and* g_1 *are arbitrary (so* S *is not one-to-one).*

Part 2: Describe all eigenvectors of S, i.e. nonzero polynomials such that $g''(x) = \lambda g(x)$ for some λ.

Answer: Since the degree of g'' *is less than the degree of* g, g'' *cannot be a nonzero multiple of* g. *Therefore* $\lambda = 0$ *and* $g''(x) = 0$ *is satisfied by all linear polynomials* $g(x) = g_0 + g_1 x$.

Question 2. (10 points) (version 1) Let A be an m-by-n complex matrix, and let B be an n-by-m complex matrix. Show that $I_m + A \cdot B$ is invertible if and only if $I_n + B \cdot A$ is invertible.

Answer: Solution 1: Suppose $I_m + A \cdot B$ is invertible, and $(I_n + B \cdot A) \cdot v = 0$; we need to show $v = 0$. Multiply by A to get

$$A \cdot (I_n + B \cdot A) \cdot v = A \cdot v + A \cdot B \cdot A \cdot v = (I_m + A \cdot B) \cdot (A \cdot v)$$

so $A \cdot v = 0$ since $I_m + A \cdot B$ is invertible. But then $v = -B \cdot A \cdot v = 0$ as desired. Thus $I_m + A \cdot B$ invertible implies $I_n + B \cdot A$ invertible. The converse follows by the same argument.

Solution 2: From the practice final, we know $A \cdot B$ and $B \cdot A$ have the identical nonzero eigenvalues. Therefore, -1 is an eigenvalue of $A \cdot B$ if and only if it an eigenvalue of $B \cdot A$, implying 0 is an eigenvalue of $I_m + A \cdot B$ if and only if it is an eigenvalue of $I_n + B \cdot A$, so that $I_m + A \cdot B$ is singular if and only if $I_n + B \cdot A$ is singular.

Question 2. (10 points) (version 2) Let X be an m-by-n real matrix, and let Y be an n-by-m complex matrix. Show that $I_m + X \cdot Y$ is invertible if and only if $I_n + Y \cdot X$ is invertible.

Answer: Solution 1: Suppose $I_m + X \cdot Y$ is invertible, and $(I_n + Y \cdot X) \cdot v = 0$; we need to show $v = 0$. Multiply by X to get

$$X \cdot (I_n + Y \cdot X) \cdot v = X \cdot v + X \cdot Y \cdot X \cdot v = (I_m + X \cdot Y) \cdot (X \cdot v)$$

so $X \cdot v = 0$ since $I_m + X \cdot Y$ is invertible. But then $v = -Y \cdot X \cdot v = 0$ as desired. Thus $I_m + X \cdot Y$ invertible implies $I_n + Y \cdot X$ invertible. The converse follows by the same argument.

Solution 2: From the practice final, we know $X \cdot Y$ and $Y \cdot X$ have the identical nonzero eigenvalues. Therefore, -1 is an eigenvalue of $X \cdot Y$ if and only if it an eigenvalue of $Y \cdot X$, implying 0 is an eigenvalue of $I_m + X \cdot Y$ if and only if it is an eigenvalue of $I_n + Y \cdot X$, so that $I_m + X \cdot Y$ is singular if and only if $I_n + Y \cdot X$ is singular.

Question 3. (10 points) (version 1) Let $A = P_R \cdot L \cdot U \cdot P_C$ be an LU decomposition of the m-by-n real matrix A of rank $r > 0$. Thus P_R and P_C are permutation matrices, L is m-by-r and unit lower triangular, and U is r-by-n and upper triangular with U_{ii} nonzero. Show how to express an LU decomposition of A^t using simple modifications of the parts of this LU decomposition of A.

Answer: Write $U = D \cdot \hat{U}$, where D is r-by-r and diagonal with $D_{ii} = U_{ii}$, so $\hat{U}$ is unit upper triangular. Then $A = P_R \cdot L \cdot U \cdot P_C = P_R \cdot L \cdot D \cdot \hat{U} \cdot P_C$ so $A^t = P_C^t \cdot \hat{U}^t \cdot (D \cdot L^t) \cdot P_R^t$. This is an LU decomposition of A^t.

Question 3. (10 points) (version 2) Let $B = P_R \cdot L \cdot U \cdot P_C$ be an LU decomposition of the m-by-n complex matrix B of rank $r > 0$. Thus P_R and P_C are permutation matrices, L is m-by-r and unit lower triangular, and U is r-by-n and upper triangular with U_{ii} nonzero. Show how to express an LU decomposition of B^* using simple modifications of the parts of this LU decomposition of B.

Answer: Write $U = D \cdot \hat{U}$, where D is r-by-r and diagonal with $D_{ii} = U_{ii}$, so $\hat{U}$ is unit upper triangular. Then $B = P_R \cdot L \cdot U \cdot P_C = P_R \cdot L \cdot D \cdot \hat{U} \cdot P_C$ so $B^* = P_C^* \cdot \hat{U}^* \cdot (\bar{D} \cdot L^*) \cdot P_R^*$. This is an LU decomposition of B^*.

Question 4. (10 points) (version 1) Let $T : V \to V$ be a linear operator. Suppose that $T(v_i) = \lambda_i v_i$ for $i = 1, ...m$, and all λ_i are distinct. If W is an invariant subspace of T and includes the vector $\sum_{i=1}^{m} a_i \cdot v_i$, where all the $a_i \neq 0$, then prove that W contains v_i for $i = 1, ..., m$.

Answer: Let $w_1 = \sum_{i=1}^{m} a_i \cdot v_i$. *Since* $w_1 \in W$, *so are* $w_2 = T \cdot w_1$ *through* $w_n = T^{n-1} \cdot w_1$ *since* W *is invariant. Let* $\tilde{V} = [v_1, ..., v_m]$ *and* $\tilde{W} = [w_1, ..., w_m]$. *Let* $A_{ij} = a_i \cdot \lambda_i^{j-1}$ *be m-by-m. Then we can express the dependence of all the* w_i *on all the* v_i *by* $\tilde{W} = \tilde{V} \cdot A$. *Now* $A = \mathrm{diag}(a_1, a_2, ..., a_n) \cdot B$, *where* $B_{ij} = \lambda_i^{j-1}$. *Thus* B *is the* m *by* m *Vandermonde matrix with distinct* λ_i, *and so nonsingular by homework question 4.3.22(c). Thus* A *is the product of nonsingular matrices and also nonsingular. Thus* $\tilde{W} \cdot A^{-1} = \tilde{V}$, *so all the columns of* $\tilde{V}$, *namely the* v_i, *are linear combinations of the* w_i, *and so within* W *as desired.*

Question 4. (10 points) (version 2) Let $S : X \to X$ be a linear operator. Suppose that $S(x_i) = \lambda_i x_i$ for $i = 1, ...m$, and all λ_i are distinct. If Y is an invariant subspace of S and includes the vector $\sum_{i=1}^{m} b_i \cdot x_i$, where all the $b_i \neq 0$, then prove that Y contains x_i for $i = 1, ..., m$.

Answer: Let $y_1 = \sum_{i=1}^{m} b_i \cdot x_i$. *Since* $y_1 \in Y$, *so are* $y_2 = S \cdot y_1$ *through* $y_n = S^{n-1} \cdot y_1$ *since* Y *is invariant. Let* $\tilde{X} = [x_1, ..., x_m]$ *and* $\tilde{Y} = [y_1, ..., y_m]$. *Let* $A_{ij} = b_i \cdot \lambda_i^{j-1}$ *be m-by-m. Then we can express the dependence of all the* y_i *on all the* x_i *by* $\tilde{Y} = \tilde{X} \cdot A$. *Now* $A = \mathrm{diag}(b_1, b_2, ..., b_n) \cdot B$, *where* $B_{ij} = \lambda_i^{j-1}$. *Thus* B *is the* m *by* m *Vandermonde matrix with distinct* λ_i, *and so nonsingular by homework question 4.3.22(c). Thus* A *is the product of nonsingular matrices and also nonsingular. Thus* $\tilde{Y} \cdot A^{-1} = \tilde{X}$, *so all the columns of* $\tilde{X}$, *namely the* x_i, *are linear combinations of the* y_i, *and so within* Y *as desired.*

Question 5. (10 points) (version 1) Let $A = X \cdot \Lambda \cdot X^{-1}$ be diagonalizable (Λ is diagonal). Let $X = QR$ be the QR decomposition of X, so that Q is unitary and R upper triangular. Show that $T = Q^* \cdot A \cdot Q$ is upper triangular. What is the name we gave to the matrix factorization $A = Q \cdot T \cdot Q^*$?

Answer: $A = X \cdot \Lambda \cdot X^{-1} = Q \cdot R \cdot \Lambda \cdot R^{-1} \cdot Q^*$, *since* $Q^{-1} = Q^*$, *and so* $Q^{-1} \cdot A \cdot Q = R \cdot \Lambda \cdot R^{-1} = T$. *Since* R *is upper triangular and nonsingular, so is* R^{-1}, *and so the product* $R \cdot \Lambda \cdot R^{-1}$ *is a product of upper triangular matrices and so upper triangular as desired. This is the Schur decomposition.*

Question 5. (10 points) (version 2) Let $B = Z \cdot \Lambda \cdot Z^{-1}$ be diagonalizable (Λ is diagonal). Let $Z = QR$ be the QR decomposition of Z, so that Q is unitary and R upper triangular. Show that $T = Q^* \cdot B \cdot Q$ is upper triangular. What is the name we gave to the matrix factorization $B = Q \cdot T \cdot Q^*$?

Answer: $B = Z \cdot \Lambda \cdot Z^{-1} = Q \cdot R \cdot \Lambda \cdot R^{-1} \cdot Q^*$, *since* $Q^{-1} = Q^*$, *and so* $Q^{-1} \cdot B \cdot Q = R \cdot \Lambda \cdot R^{-1} = T$. *Since* R *is upper triangular and nonsingular, so is* R^{-1}, *and so the product* $R \cdot \Lambda \cdot R^{-1}$ *is a product of upper triangular matrices and so upper triangular as desired. This is the Schur Factorization.*

Question 6. (10 points) (version 1) Let $< x, y > = \sum_{i=1}^{m} x_i y_i = x^t \cdot y$ be the standard dot product on $\mathbb{R}^m$. An m-by-m real symmetric matrix $T = T^t$ is called *positive definite* if $< T \cdot x, x >$ is positive for all nonzero vectors $x \in \mathbb{R}^m$.

We define the function $< \cdot, \cdot >_T : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ by

$$< x, y >_T \;=\; < Tx, y > \quad .$$

Show that $< x, y >_T$ is an inner product on $\mathbb{R}^m$ if and only if T is symmetric and positive definite.

Answer: First assume T is symmetric and positive definite: We need to confirm that $< \cdot, \cdot >_T$ satisfies the axioms of an inner product:

$$\begin{aligned} < \alpha_1 x_1 + \alpha_2 x_2, y >_T &= < T(\alpha_1 x_1 + \alpha_2 x_2), y > \\ &= < \alpha_1 T \cdot x_1 + \alpha_2 T x_2, y > \\ &= \alpha_1 < T \cdot x_1, y > + \alpha_2 < T \cdot x_2, y > \\ &= \alpha_1 < x_1, y >_T + \alpha_2 < x_2, y >_T \end{aligned}$$

as desired. Next

$$< x, y >_T \;=\; x^t \cdot T \cdot y \;=\; (x^t \cdot T \cdot y)^t \;=\; y^t \cdot T^t \cdot x \;=\; y^t \cdot T \cdot x = < y, x >_T$$

Finally, $x \neq 0$ implies $< x, x >_T \;=\; < T \cdot x, x > \;>\; 0$ as desired.

Now assume $< x, y >_T = y^t \cdot T \cdot x$ is an inner product. Then

$$T_{ij} \;=\; e_i^t \cdot T \cdot e_j \;=\; < e_i, e_j >_T \;=\; < e_j, e_i >_T \;=\; e_j^t \cdot T \cdot e_i \;=\; T_{ji}$$

so T is symmetric. Also $x \neq 0$ implies $0 \;<\; < x, x >_T \;=\; x^t \cdot T \cdot x$, so T is positive definite.

Question 6. (10 points) (version 2) Let $< u, v > = \sum_{i=1}^{n} u_i v_i = u^t \cdot v$ be the standard dot product on $\mathbb{R}^n$. An n-by-n real symmetric matrix $X = X^t$ is called *positive definite* if $< X \cdot u, u >$ is positive for all nonzero vectors $u \in \mathbb{R}^n$.

We define the function $< \cdot, \cdot >_X : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$< u, v >_X \;=\; < Xu, v >$$

Show that $< u, v >_X$ is an inner product on $\mathbb{R}^n$ if and only if X is symmetric and positive definite.

Answer: First assume X is symmetric and positive definite: We need to confirm that $< \cdot, \cdot >_X$ satisfies the axioms of an inner product:

$$\begin{aligned} < \alpha_1 u_1 + \alpha_2 u_2, v >_X &= < X(\alpha_1 u_1 + \alpha_2 u_2), v > \\ &= < \alpha_1 X \cdot u_1 + \alpha_2 X u_2, v > \\ &= \alpha_1 < X \cdot u_1, v > + \alpha_2 < X \cdot u_2, v > \\ &= \alpha_1 < u_1, v >_X + \alpha_2 < u_2, v >_X \end{aligned}$$

as desired. Next

$$< u, v >_X \;=\; u^t \cdot X \cdot v \;=\; (u^t \cdot X \cdot v)^t \;=\; v^t \cdot X^t \cdot u \;=\; v^t \cdot X \cdot u \;=\; < v, u >_X$$

Finally, $u \neq 0$ implies $< u, u >_X \;=\; < X \cdot u, u > \;>\; 0$ as desired.

Now assume $< u, v >_X \;=\; v^t \cdot X \cdot u$ is an inner product. Then

$$X_{ij} \;=\; e_i^t \cdot X \cdot e_j \;=\; < e_i, e_j >_X \;=\; < e_j, e_i >_X \;=\; e_j^t \cdot X \cdot e_i \;=\; X_{ji}$$

so X is symmetric. Also $u \neq 0$ implies $0 \;<\; < u, u >_X \;=\; u^t \cdot X \cdot u$, so X is positive definite.

Question 7. (10 points) (version 1) Find a matrix B such that $B^2 = A = \begin{bmatrix} 10 & 6 & 6 \\ 6 & 10 & 9 \\ 0 & 0 & 1 \end{bmatrix}$. What are all possible sets of eigenvalues of B?

Answer: The squares of the eigenvalues of B are the eigenvalues of A, so they can be $\{\pm 1, \pm 2, \pm 3\}$, where each $\pm$ sign can be chosen independently. Thus there are $2^3 = 8$ possible triples in all for all possible choices of signs.

One square root of A, with positive eigenvalues, is $B = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. We can compute it in (at least) two ways.

First way: Diagonalize $A = V \cdot \Lambda \cdot V^{-1}$ with $\Lambda = \mathrm{diag}(16, 4, 1)$, $V = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $V^{-1} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Then $B = V \cdot \Lambda^{1/2} \cdot V^{-1} = V \cdot \mathrm{diag}(4, 2, 1) \cdot V^{-1}$.

Second way: Write $A = \begin{bmatrix} \hat{A} & \begin{bmatrix} 6 \\ 9 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \hat{B} & \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} \\ 0 & b_{33} \end{bmatrix}$ where $\hat{A}$ and $\hat{B}$ are 2-by-2 submatrices, and then note that

$$B \cdot B = \begin{bmatrix} \hat{B}^2 & (\hat{B} + b_{33} \cdot I_2) \cdot \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} \\ 0 & b_{33}^2 \end{bmatrix}$$

Thus we see that $\hat{B}^2 = \hat{A}$. We solve for $\hat{B}$ by diagonalizing $\hat{A} = \hat{U} \cdot \hat{\Lambda} \cdot \hat{U}^{-1}$ where $\hat{U} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\hat{\Lambda} = \mathrm{diag}(16, 4)$, so $\hat{B} = \hat{U} \cdot \mathrm{diag}(4, 2) \cdot \hat{U}^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Then $b_{33}^2 = 1$ so we take $b_{33} = 1$. Finally, we solve $(\hat{B} + I) \cdot \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ for $b_{13} = 1$ and $b_{23} = 2$ as desired.

The set of all possible answers consists of

$$\pm \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 7. (10 points) (version 2) Find a matrix X such that $X^2 = Y = \begin{bmatrix} 13 & 12 & 7 \\ 12 & 13 & 7 \\ 0 & 0 & 4 \end{bmatrix}$. What are all possible sets of eigenvalues of X?

Answer: The squares of the eigenvalues of X are the eigenvalues of Y, so they can be $\{\pm 1, \pm 2, \pm 5\}$, where each $\pm$ sign can be chosen independently. Thus there are $2^3 = 8$ possible triples in all for all possible choices of signs.

One square root, with positive eigenvalues, is $X = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. We can compute it in (at least) two ways.

First way: Diagonalize $Y = V \cdot \Lambda V^{-1}$ with $\Lambda = \mathrm{diag}(25, 1, 4)$, $V = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, and $V^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$. Then $X = V \cdot \Lambda^{1/2} \cdot V^{-1} = V \cdot \mathrm{diag}(5, 1, 2) \cdot V^{-1}$.

Second way: Write $Y = \begin{bmatrix} \hat{Y} & \begin{bmatrix} 7 \\ 7 \end{bmatrix} \\ 0 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} \hat{X} & \begin{bmatrix} x_{13} \\ x_{23} \end{bmatrix} \\ 0 & x_{33} \end{bmatrix}$ where $\hat{Y}$ and $\hat{X}$ are 2-by-2 submatrices, and then note that

$$X \cdot X = \begin{bmatrix} \hat{X}^2 & (\hat{X} + x_{33} \cdot I_2) \cdot \begin{bmatrix} x_{13} \\ x_{23} \end{bmatrix} \\ 0 & x_{33}^2 \end{bmatrix}$$

Thus we see that $\hat{X}^2 = \hat{Y}$. We solve for $\hat{X}$ by diagonalizing $\hat{Y} = \hat{U} \cdot \hat{\Lambda} \cdot \hat{U}^{-1}$ where $\hat{U} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\hat{\Lambda} = \mathrm{diag}(25, 1)$, so $\hat{X} = \hat{U} \cdot \mathrm{diag}(5, 1) \cdot \hat{U}^{-1} = \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}$. Then $x_{33}^2 = 4$ so we take $x_{33} = 2$. Finally, we solve $(\hat{X} + 2 \cdot I) \cdot \begin{bmatrix} x_{13} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$ for $x_{13} = 1$ and $x_{23} = 1$ as desired.

The set of all possible answers consists of

$$\pm \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \pm \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \pm \begin{bmatrix} 3 & 2 & 7/3 \\ 2 & 3 & 7/3 \\ 0 & 0 & -2 \end{bmatrix}, \pm \begin{bmatrix} 2 & 3 & 7/3 \\ 3 & 2 & 7/3 \\ 0 & 0 & -2 \end{bmatrix}$$

Question 8. (10 points) (version 1) Determine all possible Jordan canonical forms for matrices with the characteristic polynomial $(x-1)^3(x-2)^2$. In other words, list the sets of Jordan blocks that could appear in each Jordan canonical form.

Answer: Let $J_i(\lambda)$ denote an i-by-i Jordan block with eigenvalue λ. For the triple eigenvalue at 1, the possible Jordan structures are $\{J_3(1), (J_2(1), J_1(1)), (J_1(1), J_1(1), J_1(1))\}$. for the double eigenvalue at 2, the possible Jordan structures are $\{J_2(2), (J_1(2), J_1(2))\}$. Since the Jordan structures for the 2 eigenvalues can be chosen independently, and the order in which they appear does not matter, there are 6 possible structures in all, for all choices from the first set and the second set.

Question 8. (10 points) (version 2) Determine all possible Jordan canonical forms for matrices with the characteristic polynomial $(x-3)^2(x-4)^3$. In other words, list the sets of Jordan blocks that could appear in each Jordan canonical form.

Answer: Let $J_i(\lambda)$ denote a i-by-i Jordan block with eigenvalue λ. For the triple eigenvalue at 4, the possible Jordan structures are $\{J_3(4), (J_2(4), J_1(4)), (J_1(4), J_1(4), J_1(4))\}$. for the double eigenvalue at 3, the possible Jordan structures are $\{J_2(3), (J_1(3), J_1(3))\}$. Since the Jordan structures for the 2 eigenvalues can be chosen independently, and the order in which they appear does not matter, there are 6 possible structures in all, for all choices from the first set and the second set.