Towards accurate polynomial evaluation

or

When can Numerical Linear Algebra be done accurately?

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Joint work with
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Outline

1. Motivation and goal(s).
2. Model of arithmetic and setting.
3. What is *allowable* in classical arithmetic.
4. Results for classical arithmetic, real and complex.
5. What is *allowable* in black-box arithmetic.
6. Results for black-box arithmetic, real and complex.
7. Other models of arithmetic.
8. Open problems / Future work.

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Goal

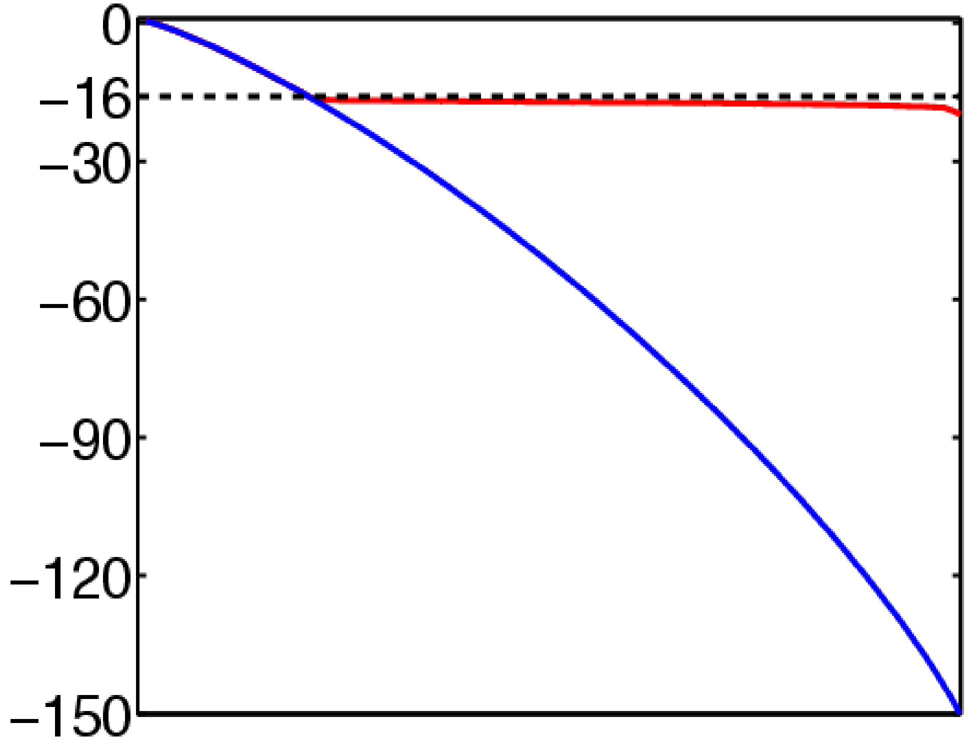
Given a family of structured matrices $M(x)$,
find **accurate** and **efficient** algorithms
to solve linear algebra problems (eg $y = \det M(x)$ or $y = \text{eig}(M(x))$),
or prove that none exist

Accurately means relative error $\eta < 1$, i.e.

- $|y_{\text{computed}} - y| \leq \eta\ |y|$,
- $\eta = 10^{-2}$ yields two digits of accuracy,
- $y_{\text{computed}} = 0 \iff y = 0$.

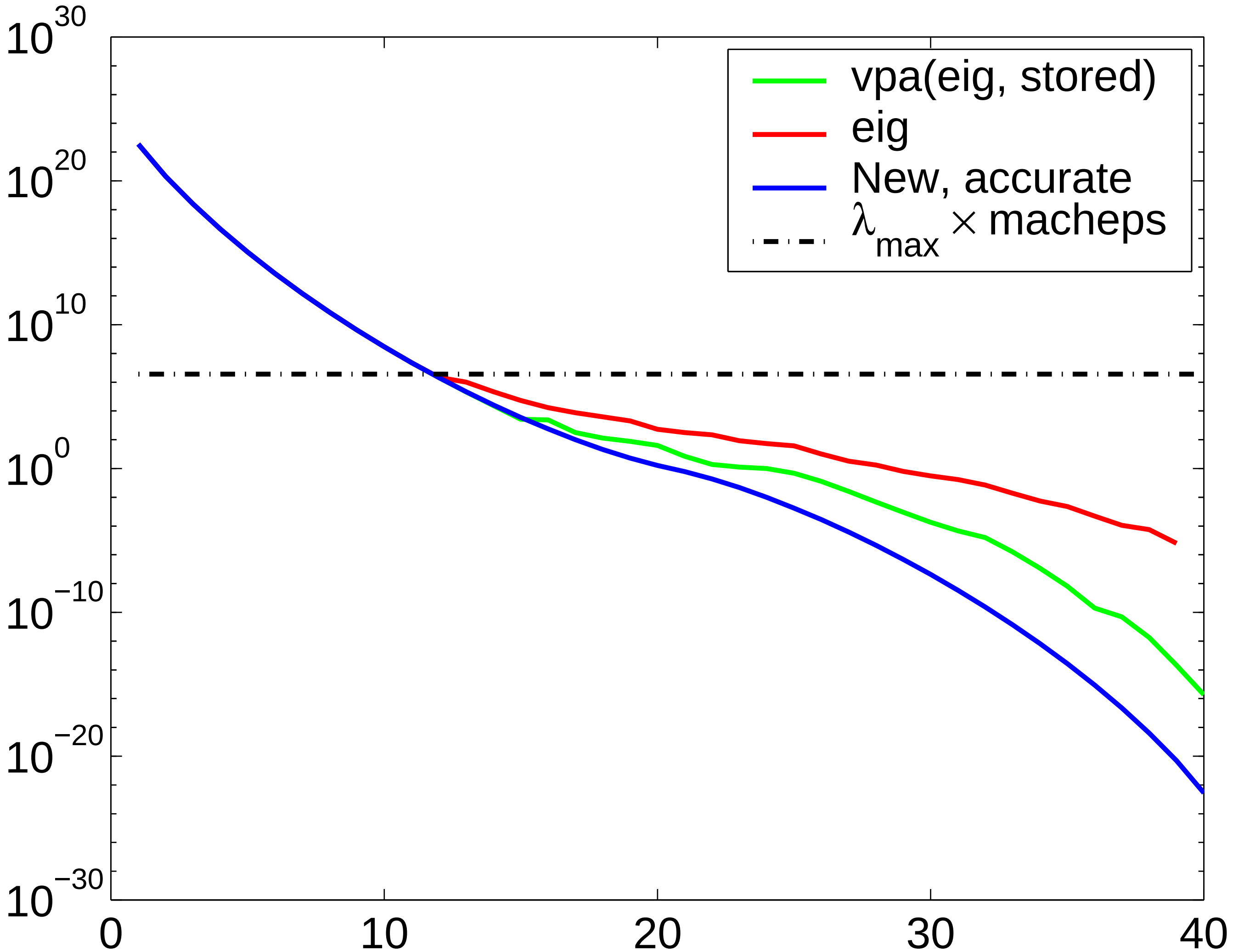
Efficiently means in polynomial time

$\mathbf{Log_{10}}$(Eigenvalues) of 50x50 Hilbert Matrix

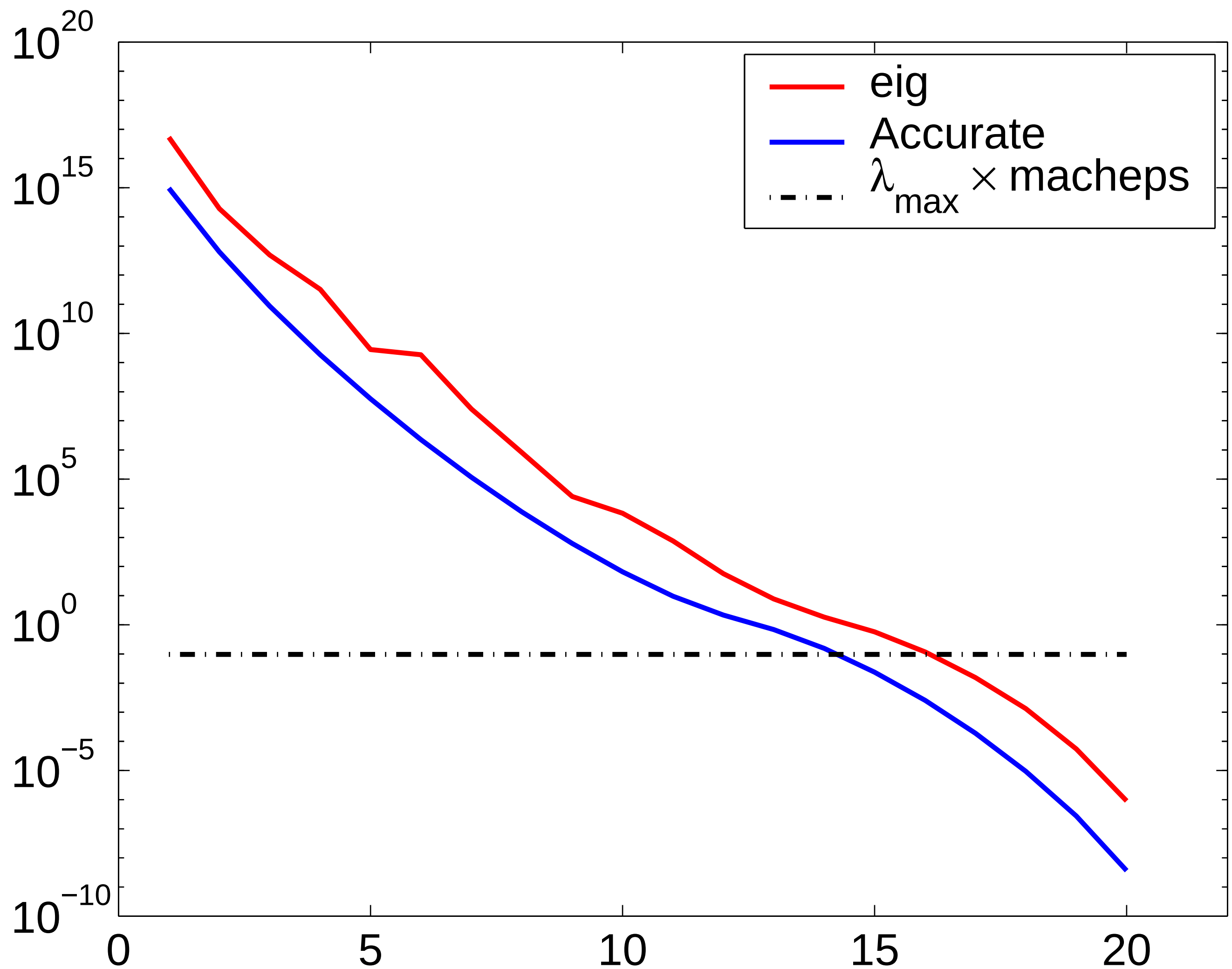


red line shows eigenvalues from conventional algorithm in 16 digits
blue line shows eigenvalues from new algorithm in 16 digits
Cost of guaranteed accuracy: $O(n^3 \log \kappa)$ vs $O(n^3 \log \log \kappa)$ where
κ = condition number

Eigenvalues of 40x40 Pascal Matrix



Eigenvalues of 20x20 Schur complement of 40x40 Vandermonde Matrix



None correct vs All correct!

General Structured Matrices

Type of matrix		det A	A^{-1}	Any minor	LDU	SVD	Sym EVD
Acyclic (bidiagonal and other)							
Total Sign Compound (TSC)							
Diagonally Scaled Totally Unimodular (DSTU)							
Weakly diagonally dominant M-matrix							
Displacement Rank One	Cauchy						
	Vandermonde						
	Polynomial Vandermonde						
Toeplitz							

General Structured Matrices

Type of matrix		det A	A^{-1}	Any minor	LDU	SVD	Sym EVD
Acyclic (bidiagonal and other)		n	n^2	n	$\leq n^2$	n^3	N/A
Total Sign Compound (TSC)		n	n^3	n	n^4	n^4	n^4
Diagonally Scaled Totally Unimodular (DSTU)		n^3	n^5?	n^3	n^3	n^3	n^3
Weakly diagonally dominant M-matrix		n^3	n^3	?	n^3	n^3	n^3
Displace-ment Rank One	Cauchy	n^2	n^2	n^2	$\leq n^3$	n^3	n^3
	Vandermonde	n^2	?	?	?	n^3	n^3
	Polynomial Vandermonde	n^2	?	?	?	?	?
Toeplitz		?	?	?	?	?	?

Totally Nonnegative Matrices

Type of Matrix	det A	A^{-1}	Any minor	Gauss. elim. NP	PP	CP	NE NP	Ax=b	SVD	Eig. Val.
Cauchy										
Vandermonde										
Generalized Vandermonde										
Any TN in Neville form										

Totally Nonnegative Matrices

Type of Matrix	det A	A^{-1}	Any minor	Gauss. elim. NP	Gauss. elim. PP	Gauss. elim. CP	NE NP	Ax=b	SVD	Eig. Val.
Cauchy	n^2	n^2	n^2	n^2	n^3	n^3	n^2	n^2	n^3	n^3
Vandermonde	n^2	n^3	n^3	n^2	n^2	poly	n^2	n^2	n^3	n^3
Generalized Vandermonde	n^2	n^3	poly	n^2	n^2	poly	n^2	n^2	n^3	n^3
Any TN in Neville form	n	n^3	n^3	n^3	n^3	n^3	0	n^2	n^3	n^3

poly = poly(n, λ), where λ = partition

(see talk by Plamen Koev, Tuesday 4pm)

Reduce Matrix problem to Polynomial problem

Theorem: Being able to compute $\det(M)$ accurately is *necessary* to be able to compute LDU, eig, SVD, ... accurately

Theorem: Being able to compute all minors of M accurately is *sufficient* for computing M^{-1}, LDU, SVD, ... accurately

(Sufficient conditions for computing eig(M) accurately unknown in nonsymmetric, non-totally positive case)

Goal - restated

Given a polynomial (or a family of polynomials) p, either produce an **accurate** algorithm to compute $y = p(x)$, or prove that none exists.

Accurately means relative error $\eta < 1$, i.e.

- $|y_{\text{computed}} - y| \leq \eta\, |y|$,
- $\eta = 10^{-2}$ yields two digits of accuracy,
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Traditional Model of Arithmetic

- $fl(a \otimes b) = (a \otimes b)(1+\delta)$, with arbitrary roundoff error $|\delta| < \epsilon \ll 1$
  - a, b and δ all real, or all complex
- Operations?

Traditional Model of Arithmetic

- $fl(a \otimes b) = (a \otimes b)(1+\delta)$, with arbitrary roundoff error $|\delta| < \epsilon \ll 1$
- Operations?
  - in classical arithmetic, $+$, $-$, $\times$; also exact negation;

Traditional Model of Arithmetic

- $fl(a \otimes b) = (a \otimes b)(1 + \delta)$, with arbitrary roundoff error $|\delta| < \epsilon \ll 1$
- Operations?
  - in classical arithmetic, $+$, $-$, $\times$; also exact negation;
  - How can we lose accuracy in this model?
    * OK to multiply or add positive numbers
    * OK to subtract exact numbers (initial data)
    * Accuracy may only be lost when subtracting approximate results:

$$
\begin{array}{rl}
 & .12345xxx \\
- & .12345yyy \\
\hline
 & .00000zzz
\end{array}
$$

Recognizing Accuracy

- **Ex: Compute** $p(x) = x_1 + x_2 + x_3$
  - **Try** $alg(x,\delta) = ((x_1 + x_2)(1+\delta_1) + x_3)(1+\delta_2)$

$$
\begin{aligned}
rel\_err(x,\delta) &= \frac{alg(x,\delta) - p(x)}{p(x)} \\
&= \frac{x_1 + x_2}{x_1 + x_2 + x_3}(\delta_1 + \delta_2 + \delta_1 \cdot \delta_2) + \frac{x_3}{x_1 + x_2 + x_3}(\delta_2)
\end{aligned}
$$

  - $\forall \epsilon > 0$, $rel\_err(x,\delta)$ **unbounded on an open subset of** (x,δ) **with** $|\delta_i| < \epsilon$
- **Generally:** $rel\_err(x,\delta) = \sum_r \frac{p_r(x)}{p(x)} \cdot q_r(\delta)$
  - **Each** $\frac{p_r(x)}{p(x)}$ **must be bounded near** $p(x) = 0$
- **Ex:** $p(x)$ **positive definite and homogeneous, degree** d
  - **If** $p_r(x)$ **also homogeneous, degree** d**, then** $\frac{p_r(x)}{p(x)}$ **bounded**

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  - in classical arithmetic, $+$, $-$, $\times$; also exact negation;
  - in black-box arithmetic, above plus selected polynomial expressions
    - Ex: $x - yz$ (IBM's fused-multiply-add)
    - Ex: $wx - yz$ (using double-double)
    - Ex: small determinants (Shewchuk's Triangle)
    - Ex: dot products (using Priest or Demmel/Hida algs)

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- Constants?

Availability of constants?

Example.

- Classical case:
  - without $\sqrt{2}$, we cannot compute

$$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$$

    accurately.
  – having no explicit constants no loss of generality for homogeneous, integer-coefficient polynomials.
- Black-box case:
  - any constants we choose can be accommodated via black-boxes

Traditional Model of Arithmetic.

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  - exact answer in finite # of steps in absence of roundoff error

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  - branching based on comparisons
  - non-determinism (because determinism is simulable)

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- Algorithms?
  - exact answer in finite # of steps in absence of roundoff error
  - branching based on comparisons
  - non-determinism (because determinism is simulable)
  - domains to be $\mathbb{C}^n$ or $\mathbb{R}^n$ (but some domain-specific results).

Problem Restatement

◇ Notation:

– $p(x)$ multivariate polynomial to be evaluated, $x = (x_1, \ldots, x_k)$.

– $\delta = (\delta_1, \ldots, \delta_m)$ is the vector of error (rounding) variables.

– $p_{comp}(x, \delta)$ is the result of algorithm to compute p at x with errors δ.

◇ Goal: Decide if $\exists$ algorithm $p_{comp}(x, \delta)$ to accurately evaluate $p(x)$ on $\mathcal{D}$:

$\forall\ 0 < \eta < 1$ ... for any η = desired relative error

$\exists\ 0 < \epsilon < 1$ ... there is an ϵ = maximum rounding error

$\forall\ x \in \mathcal{D}$ ... so that for all x in the domain

$\forall\ |\delta_i| \leq \epsilon$ ... and for all rounding errors bounded by ϵ

$|p_{comp}(x, \delta) - p(x)| \leq \eta \cdot |p(x)|$ ... relative error is at most η

◇ Given $p(x)$ and $\mathcal{D}$, seek effective procedure ("compiler") to exhibit algorithm, or show one does not exist

Examples in classical arithmetic over $\mathbb{R}^n$ (none work over $\mathbb{C}^n$).

- $M_2(x,y,z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 2 \cdot z^2)$
  - Positive definite and homogeneous, easy to evaluate accurately
- $M_3(x,y,z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 3 \cdot z^2)$
  - Motzkin polynomial, nonnegative, zero at $|x| = |y| = |z|$

$$
\begin{array}{rl}
\text{if} & |x-z| \le |x+z| \wedge |y-z| \le |y+z| \\
p = & z^4 \cdot [4((x-z)^2 + (y-z)^2 + (x-z)(y-z))] + \\
 & +z^3 \cdot [2(2(x-z)^3 + 5(y-z)(x-z)^2 + 5(y-z)^2(x-z) + \\
 & \qquad 2(y-z)^3)] + \\
 & +z^2 \cdot [(x-z)^4 + 8(y-z)(x-z)^3 + 9(y-z)^2(x-z)^2 + \\
 & \qquad 8(y-z)^3(x-z) + (y-z)^4] + \\
 & +z \cdot [2(y-z)(x-z)((x-z)^3 + 2(y-z)(x-z)^2 + \\
 & \qquad 2(y-z)^2(x-z) + (y-z)^3] + \\
 & +(y-z)^2(x-z)^2((x-z)^2 + (y-z)^2) \\
\text{else} & \ldots \; 2^{\#\text{vars}-1} \; \text{more analogous cases}
\end{array}
$$

- $M_4(x,y,z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 4 \cdot z^2)$
  - Impossible to evaluate accurately

Sneak Peak.

The variety,

$$V(p) = \{x : p(x) = 0\} \ ,$$

plays a necessary role.

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Allowable varieties in classical arithmetic

Define *basic allowable sets*:

- $Z_i = \{x : x_i = 0\}$,
- $S_{ij} = \{x : x_i + x_j = 0\}$,
- $D_{ij} = \{x : x_i - x_j = 0\}$.

A variety $V(p)$ is *allowable* if it can be written as a finite union of intersections of basic allowable sets.

Denote by

$$\mathbf{G}(\mathbf{p}) = \mathbf{V}(\mathbf{p}) - \cup_{\textbf{allowable } \mathbf{A} \subset \mathbf{V}(\mathbf{p})} \mathbf{A}$$

the set of points *in general position.*

$V(p)$ unallowable $\quad \Rightarrow \quad G(p) \neq \emptyset$.

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Necessary condition on $V(p)$ for accurate evaluation of p

Theorem 1: $V(p)$ unallowable $\Rightarrow p$ cannot be evaluated accurately on $\mathbb{R}^n$ or on $\mathbb{C}^n$.

Theorem 2: On a domain $\mathcal{D}$, if $\mathrm{Int}(\mathcal{D}) \cap G(p) \neq \emptyset$, p cannot be evaluated accurately.

Examples on $\mathbb{R}^n$, revisited

- $p(x, y, z) = x + y + z$ UNALLOWABLE
- $M_2(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 2 \cdot z^2)$

  ALLOWABLE, $V(p) = \{0\}$.

- $M_3(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 3 \cdot z^2)$

  ALLOWABLE, $V(p) = \{|x| = |y| = |z|\}$

- $M_4(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 4 \cdot z^2)$

  UNALLOWABLE

- V(det(Toeplitz)), UNALLOWABLE $\Rightarrow$ no accurate linear algebra for Toeplitz in classical arithmetic
- V(minor(Vandermonde)), UNALLOWABLE, but ok on positive orthant (TP matrices)

Necessary condition on $V(p)$, real and complex

Theorem 1: $V(p)$ unallowable $\Rightarrow p$ cannot be evaluated accurately on $\mathbb{R}^n$ or on $\mathbb{C}^n$.

Theorem 2: On a domain $\mathcal{D}$, if $\mathrm{Int}(\mathcal{D}) \cap G(p) \neq \emptyset$, p cannot be evaluated accurately.

Sketch of proof.

Simplest case: non-branching, no data reuse (except for inputs), non-determinism.

Algorithm can be represented as a tree with extra edges from the sources, each node corresponds to an operation $(+, -, \times)$, each node has a specific δ, each node has two inputs, one output.

Let $x \in G(p)$ and define $Allow(x)$ as the smallest allowable set containing x.

Necessary condition on $V(p)$, real and complex.

Theorem 1: $V(p)$ unallowable $\Rightarrow$ p cannot be evaluated accurately on $\mathbb{R}^n$ or on $\mathbb{C}^n$.

Theorem 2: On a domain $\mathcal{D}$, if $\mathrm{Int}(\mathcal{D}) \cap G(p) \neq \emptyset$, p cannot be evaluated accurately.

Sketch of proof, cont'd.

Key fact: for a positive measure set of δs in δ-space, a zero output can be "traced back" down the tree to "allowable" condition ($x_i = 0$ or $x_i + x_j = 0$), or trivial one ($x_i - x_i = 0$).

So for a positive measure set of δs, either

- $p_{comp}(x, \delta)$ is not 0 (though $p(x) = 0$), or
- for all $y \in Allow(x) \setminus V(p)$, $p_{comp}(y, \delta) = 0$ (though $p(y) \neq 0$).

In either case, the polynomial is not accurately evaluable arbitrarily close to x, q.e.d.

Sufficient condition on $V(p)$ for accurate evaluation of p, complex case.

Theorem. Let p be a polynomial over $\mathbb{C}^n$ with integer coefficients. If $V(p)$ is allowable, then p is accurately evaluable.

Sketch of proof.
Can write

$$p(x) = c \prod_i p_i(x) \ ,$$

where $p_i(x)$ is a power of some x_j or $x_j \pm x_k$, and c is an integer; all operations are accurate.

Sufficient Condition on $V(p)$ for accurate evaluation of p, complex case.

Theorem. Let p be a polynomial over $\mathbb{C}^n$ with integer coefficients. If $V(p)$ is allowable, then p is accurately evaluable.

Sketch of proof.
Can write

$$p(x) = c \prod_i p_i(x) \ ,$$

where $p_i(x)$ is a power of some x_j or $x_j \pm x_k$, and c is an integer; all operations are accurate.

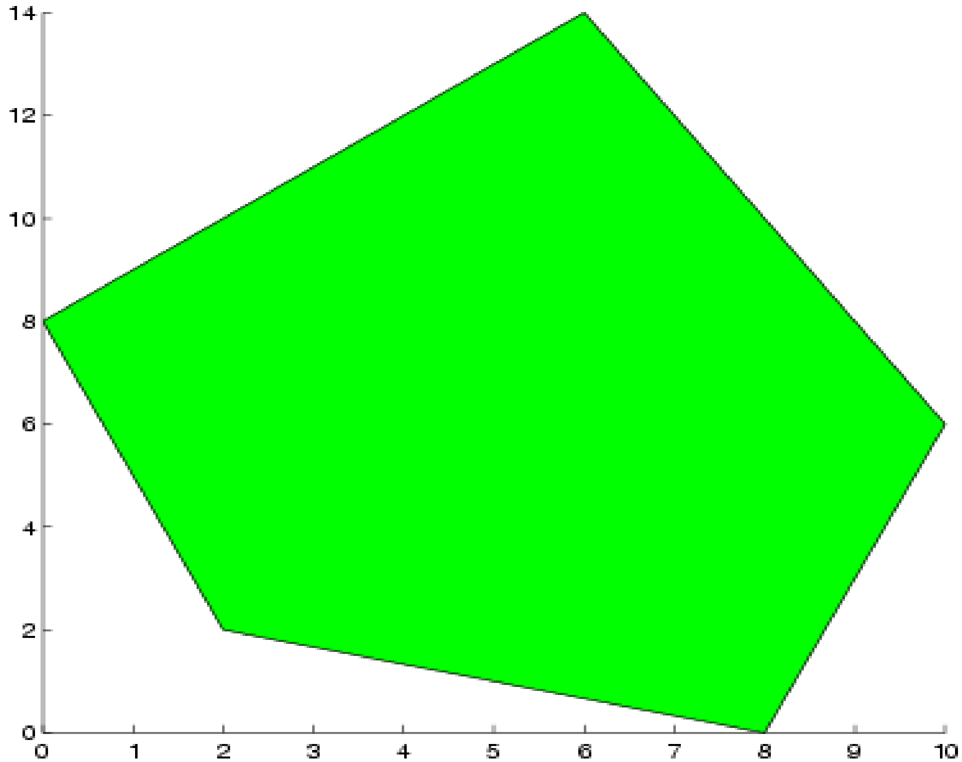
Corollary. If p is a complex multivariate polynomial, p is accurately evaluable iff p has integer coefficients and $V(p)$ is allowable.

Sufficient condition for accurate evaluation, real case.

Trickier... Allowability (or any condition) of $V(p)$ *not* sufficient:

- $q = (u^4 + v^4) + (u^2 + v^2)(x^2 + y^2 + z^2)$, $V(p) = \{u = v = 0\}$: allowable and accurately evaluable
- $p = (u^4 + v^4) + (u^2 + v^2)(x + y + z)^2$, $V(p) = \{u = v = 0\}$: allowable but NOT accurately evaluable!
- Say $p = (u^4 + v^4) + (u^2 + v^2)\hat{p}$ is "locally dominated" by $\hat{p}$ near $V(p)$
  - Accurate evaluabilty of p depends on that of $\hat{p}$
  - Leads to induction on hierarchy of varieties and polynomials defined by "dominance"
  - Need to formally define dominance
  - Induction is work in progress

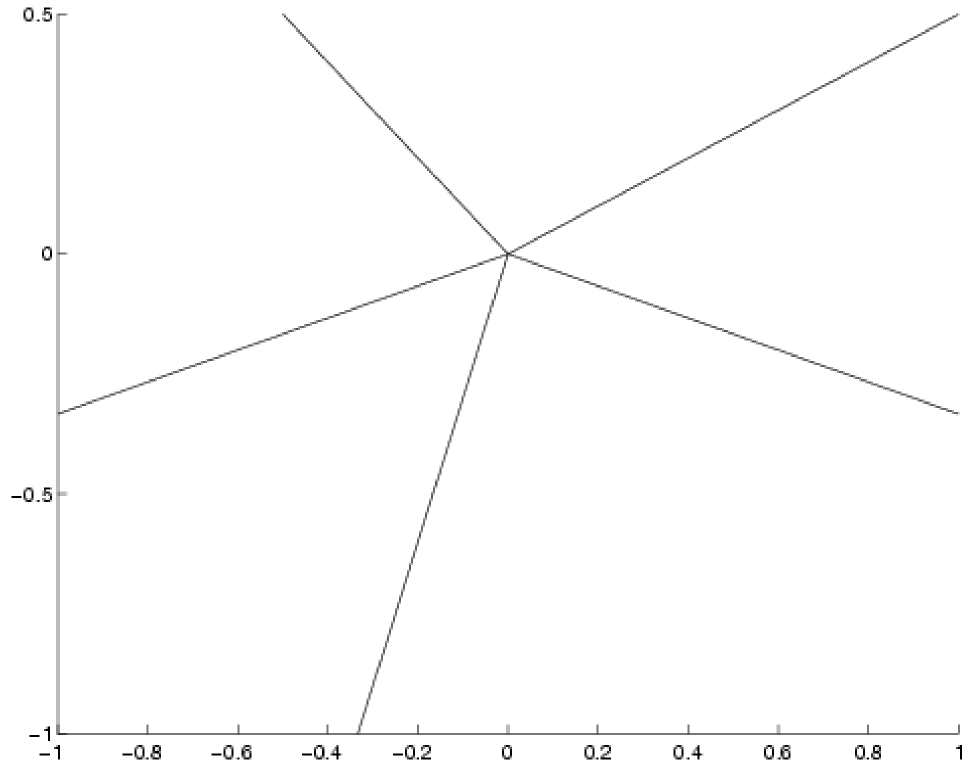
What is Dominance? Newton Polytope



$$p(x,y,z) = y^8z^{12} + x^2y^2z^{16} + x^8z^{12} + x^6y^{14} + x^{10}y^6z^4$$

Component of $V(p)$ where $\{x = y = 0\}$

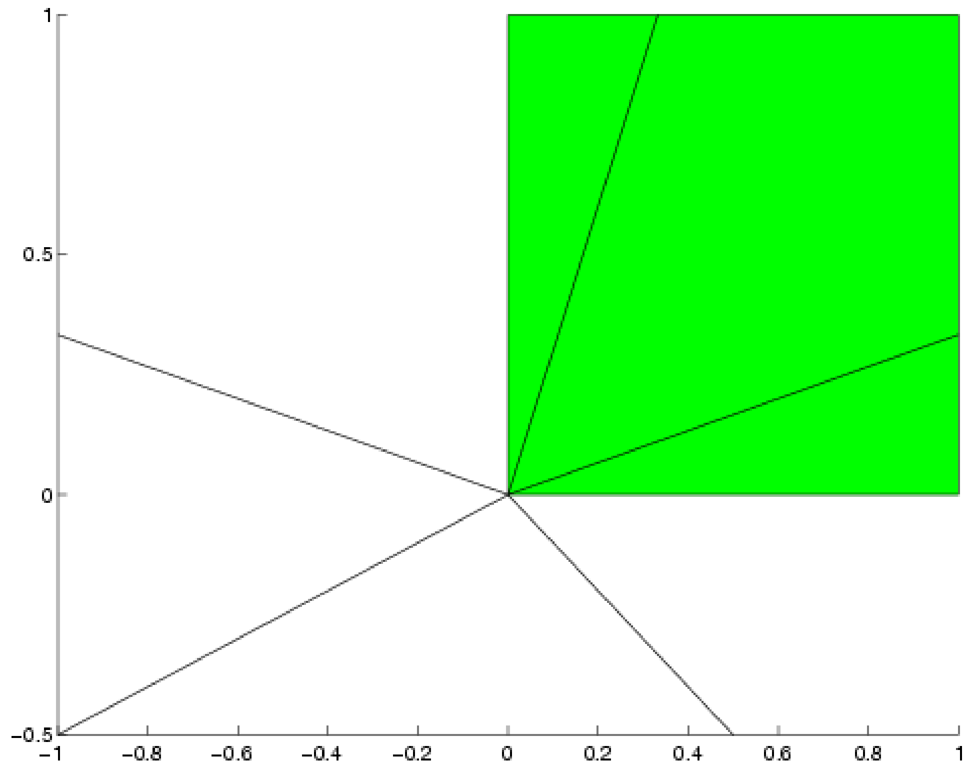
What is Dominance? Normal Fan



$$p(x,y,z) = y^8z^{12} + x^2y^2z^{16} + x^8z^{12} + x^6y^{14} + x^{10}y^6z^4$$

Component of $V(p)$ where $\{x = y = 0\}$

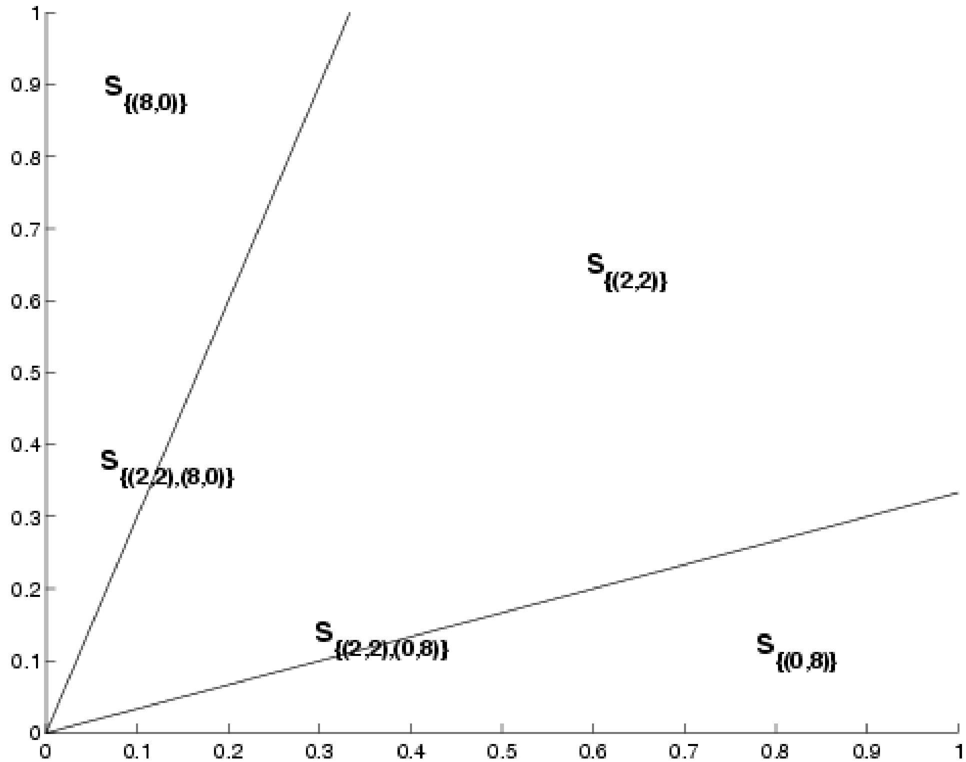
What is Dominance? First orthant of -(Normal Fan)



$$p(x, y, z) = y^8 z^{12} + x^2 y^2 z^{16} + x^8 z^{12} + x^6 y^{14} + x^{10} y^6 z^4$$

Component of $V(p)$ where $\{x = y = 0\}$

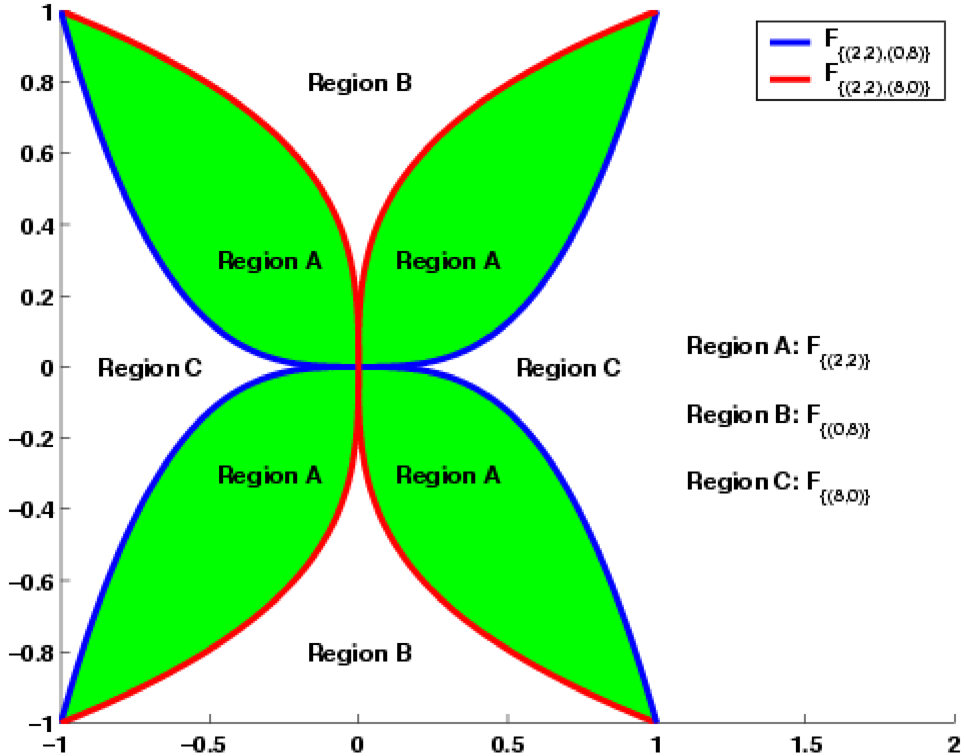
What is Dominance? Labeling cones by dominant terms



$$p(x,y,z) = y^8 z^{12} + x^2 y^2 z^{16} + x^8 z^{12} + x^6 y^{14} + x^{10} y^6 z^4$$

Component of $V(p)$ where $\{x = y = 0\}$

What is Dominance? (x, y) regions where different terms dominate - by exponentiating cones



$$p(x, y, z) = y^8z^{12} + x^2y^2z^{16} + x^8z^{12} + x^6y^{14} + x^{10}y^6z^4$$

Component of $V(p)$ where $\{x = y = 0\}$

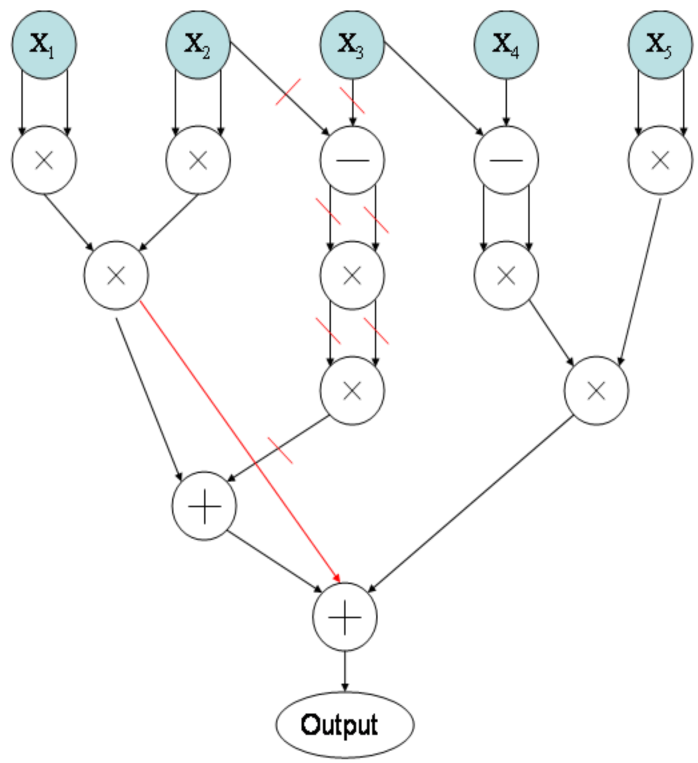
Sufficient condition for accurate evaluation, real case.

Trickier... Allowability *not* sufficient:

- $q = (u^4 + v^4) + (u^2 + v^2)(x^2 + y^2 + z^2)$, $V(p) = \{u = v = 0\}$: allowable and accurately evaluable
- $p = (u^4 + v^4) + (u^2 + v^2)(x + y + z)^2$, $V(p) = \{u = v = 0\}$: allowable but NOT accurately evaluable!
- Say $p = (u^4 + v^4) + (u^2 + v^2)\hat{p}$ is "locally dominated" by $\hat{p}$ near $V(p)$

Theorem. If all "dominant terms" are accurately evaluable on $\mathbb{R}^n$ then p is accurately evaluable. In non-branching case, if p is accurately evaluable on $\mathbb{R}^n$, then so are all "dominant terms".

Sketch of showing that accurate evaluation of dominant terms is necessary for accurate evalution of p



Pruning is used to create accurate algorithm for any dominant term from accurate algorithm for p

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Allowable varieties in black-box arithmetic

Define **black-boxes** $q_1, q_2, \ldots, q_k$ polynomial operations with various inputs, and for any j,

$$\mathcal{V}_j = \{V \neq \mathbb{R}^n : V \text{ can be obtained from } q_j \text{ through } \textbf{Process A}\text{, below}\}$$

Process A:

Step 1. repeat and/or negate, or 0 out some of the inputs,

Step 2. of the remaining variables, keep some symbolic, and find the variety in terms of the others.

Example: $q_1(x, y) = x - y$ has (up to symmetry)

$$\mathcal{V}_1 = \{\{x = 0\}, \{x - y = 0\}, \{x + y = 0\}\},$$

$q_2(x, y, z) = x - y \cdot z$ has (up to symmetry)

$$\begin{aligned}\mathcal{V}_2 = \{&\{x = 0\},\ \{y = 0\} \cup \{z = 0\},\ \{x = 0\} \cup \{x = 1\},\ \{x = 0\} \cup \{x = -1\},\\ &\{x = 0\} \cup \{y = 1\},\ \{x = 0\} \cup \{y = -1\},\ \{x - y^2 = 0\},\ \{x + y^2 = 0\},\\ &\{x - yz = 0\},\ \{x + yz = 0\}\}.\end{aligned}$$

Allowable varieties in black-box arithmetic

Define **black-boxes** $q_1, q_2, \ldots, q_k$ polynomial operations with various inputs, and for any j,

$$\mathcal{V}_j = \{V \neq \mathbb{R}^n : V \text{ can be obtained from } q_j \text{ through } \textbf{Process A}\}$$

Define *basic allowable sets*:

- $Z_i = \{x : x_i = 0\}$,
- $S_{ij} = \{x : x_i + x_j = 0\}$,
- $D_{ij} = \{x : x_i - x_j = 0\}$,
- any V for which there is a j such that $V \in \mathcal{V}_j$.

Allowable varieties in black-box arithmetic

Define **black-boxes** $q_1, q_2, \ldots, q_k$ polynomial operations with various inputs, and for any j,

$\mathcal{V}_j = \{V \neq \mathbb{R}^n : V$ can be obtained from q_j through **Process A**$\}$

A variety $V(p)$ is *allowable* if it is a union of irreducible parts of finite intersections of basic allowable sets.

Denote by

$$\mathbf{G}(\mathbf{p}) = \mathbf{V}(\mathbf{p}) - \cup_{\text{allowable } \mathbf{A} \subset \mathbf{V}(\mathbf{p})} \mathbf{A}$$

the set of points *in general position.*

$V(p)$ unallowable $\quad \Rightarrow \quad G(p) \neq \emptyset.$

Outline

1. Motivation and goal(s).
2. Model of arithmetic and setting.
3. What is *allowable* in classical arithmetic.
4. Results for classical arithmetic, real and complex.
5. What is *allowable* in black-box arithmetic.
6. Results for black-box arithmetic, real and complex.
7. Other models of arithmetic.
8. Open problems / Future work.

Necessary condition on $V(p)$ for accurate evaluation of p, real and complex

Theorem 1: $V(p)$ unallowable $\Rightarrow$ p cannot be evaluated accurately on $\mathbb{R}^n$ or on $\mathbb{C}^n$.

Theorem 2: On a domain $\mathcal{D}$, if $\text{Int}(\mathcal{D}) \cap G(p) \neq \emptyset$, p cannot be evaluated accurately.

Sufficiency condition, complex, for all q_j irreducible.

Theorem: If $V(p)$ is a union of intersections of sets Z_i, S_{ij}, D_{ij}, and $V(q_j)$, then p **is** accurately evaluable.

Corollary: If all q_j are affine, then p is accurately evaluable iff $V(p)$ is allowable.

General Structured Matrices

Type of matrix		det A	A^{-1}	Any minor	LDU	SVD	Sym EVD
Acyclic (bidiagonal and other)		n	n^2	n	$\leq n^2$	n^3	N/A
Total Sign Compound (TSC)		n	n^3	n	n^4	n^4	n^4
Diagonally Scaled Totally Unimodular (DSTU)		n^3	n^5?	n^3	n^3	n^3	n^3
Weakly diagonally dominant M-matrix		n^3	n^3	No	n^3	n^3	n^3
Displace-ment Rank One	Cauchy	n^2	n^2	n^2	$\leq n^3$	n^3	n^3
	Vandermonde	n^2	No	No	No	n^3	n^3
	Polynomial Vandermonde	n^2	No	No	No	$*$	$*$
Toeplitz		No	No	No	No	No	No

$*$ = it depends on polynomial (eg orthogonal ok)

Other linear algebra consequences

- Let $M_n(x)$ be a family of n-by-n structured matrices
- Thm: If $\det(M_n(x))$ has an irreducible factor $p_n(x)$ over $\mathbb{C}$ whose degree grows with n, then no set of "black-boxes" of bounded degree can accurately evaluate all $\det(M_n(x))$ over $\mathbb{C}$.
- Cor: $\det(\mathrm{Toeplitz}_n(x))$ cannot be evaluated accurately by any set of "black-boxes" of bounded degree over $\mathbb{C}$.
- Thm: If $V_{\mathbb{R}}(\det(M_n(x)))$ has a regular point at which the tangent depends on a growing number of coordinates, then no set of "black-boxes" of bounded degree can accurately evaluate all $\det(M_n(x))$ over $\mathbb{R}$.
- Cor: $\det(\mathrm{Toeplitz}_n(x))$ cannot be evaluated accurately by any set of "black-boxes" of bounded degree over $\mathbb{R}$.
- Accurate Toeplitz matrix computations need "infinite precision"
- What other $M_n(x)$ share these properties?

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Other Models of arithmetic

- Other models of real arithmetic
  - Blum/Shub/Smale, Cucker/Smale, Pour-El/Richards
- Comparing Reals and Integers
  - Reals, with rounded arithmetic as described
    - Some (most) $p(x)$ impossible to evaluate accurately
  - Integers, with bit operations (usual Turing machine)
    - All $p(x)$ evaluable exactly, only question is cost
    - $\det(M)$ evaluable in polynomial time
    - Not a good bit model for real arithmetic

A bit model for Reals

- $x = m \cdot 2^e$, m and e integers, with bit operations
- Still a Turing machine, but inputs better capture reals
- Models floating point arithmetic
- All $p(x)$ evaluable exactly, but cost can be much higher
- Cost of arbitrary bit of $\prod_i (1 + 2^{e_i})$ same as permanent
- Cost of $x + y + z$ exponential unless done carefully (next slide)
- Cost of $\det(M)$ unknown, even for tridiagonal
- Cost of new matrix algorithms exponentially lower than conventional algorithms to guarantee same accuracy
  - $\log \log \kappa$ vs $\log \kappa$
  - $\log \log \kappa$ is polynomial in size of input

Adding Numbers in Bit Model of Arithmetic

- $x = m \cdot 2^e$ **where** m **and** e **are integers**
- **Cancellation is obstable to accuracy:**
  - $(2^e + 1) - 2^e$ **requires** e **bits of intermediate precision**
  - **Not polynomial time in size of input** $\log_2 e$
- **"Sort and Sum" Algorithm for** $S = \sum_{i=1}^{n} x_i$

  Sort so $|e_1| \geq |e_2| \geq \cdots \geq |e_n|$ **...** $|x_1| \geq \cdots \geq |x_n|$ **more than enough**
  $S = 0$ **...** $B > b$ **bits**
  for $i = 1$ **to** n
  $\quad S = S + x_i$

- **Thm: Let** $N = 1 + 2^{B-b} + 2^{B-2b} + \cdots 2^{B \bmod b} = 1 + \lceil \frac{2^{B-b}}{1-2^{-b}} \rceil$**. Then**
  - **If** $n \leq N$**, then** S **accurate to nearly** b **bits, despite any cancellation**
  - **If** $n \geq N + 2$**, then** S **may be completely wrong (wrong sign)**
  - **If** $n = N + 1$**, in between these cases, depending on underflow**
- **Ex:** x_i **double** ($b = 53$)**,** S **extended** ($B = 64$) $\Rightarrow N = 2049$

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Open problems / Future work.

- **Complete** the decision procedure (analyze the dominant terms) when the domain is $\mathbb{R}^n$ and $V(p)$ allowable.
- **Narrow** the necessity and sufficiency conditions for the black-box case
- **Extend** to semi-algebraic domains $\mathcal{D}$.
- **Apply** to more structured matrix classes
- **Incorporate** division, rational functions, perturbation theory.
  - **Conjecture** (Demmel, '04): Accurate evaluation is possible iff condition number has only certain simple singularities (depend on reciprocal distance to set of ill-posed problems).
- **Extend** to interval arithmetic.
- **Implement** decision procedure to "compile" an accurate evaluation program given $p(x)$, $\mathcal{D}$, and minimal set of "black boxes"