PROFESSOR: So my surprise, we're on schedule. The first schedule on the web page which says which sections we're going to do on which days, we're okay on that. So what we did last time was review, I introduced derivative as the slope of a tangent line, a tangent to the curve \( Y = F(X) \). So here's my curve. I'll call the curve \( C \). And it's the graph of \( Y = F(X) \). And I have the intuition that there's a line which touched at and here's point \( P \), which, going to be the point, call this \( X \)-comma \( F(X) \). And I intuition that there's a very special line, called tangent line which touched the curve at exactly one point \( P \). Here's the points \( P \). And \( P \)-was tangent line. And as a lining it has a slope. All lines have slopes. And the slope of \( T \)-was the are defined now just sort of by your intuition, definition of driven. It's the slope of that line. So that's the picture that defined the priser. It didn't tell us how to computed it. So the way I compute it was to say I really don't know how to write down the equation for line \( T \). I could if I just pulled out the slope, I'd be done. So I draw a line through two points. To I pick a point on the curve. It was just a little weighted over. \( X \)-plus \( H \). That point is \( X \)-plus \( H \). \( F(X) \)-of \( X \)-plus \( H \). That's the point. And I draw a straight line through those two points. And the name of such a thing is called a secant. Right? If I know that this line goes through two-point I can write down the slope. I know the slope of a line if I know two points on it. The slope of secant is what? It's this distance divided by that distance. That's the slope of line. What is this distinction? distinction? Was it's the top line, how much goes from there to there. So from there to there is \( H \). So the slope of that line I have a nice simple formula is \( F(X) \)-of \( X \)-plus \( H \)-minus \( F(X) \)-of \( X \)-over \( H \). And if you have a formula for \( F \)-can write down a formula for this, so it's some amtion break thing. I think how to write that down. I don't want the slope of this curve \( S \)-. It's obviously tilted differently. Imagine we do the following. Imagine I take this point here, just a point, and slide it down the curve. You can imagine that happening. Make \( H \)-get smaller and smaller. And slide this point down. But \( S \)-keeps gowing through these two points am keep \( S \)-going through the two points. What happens to the curve \( S \)-. It starts like this. And keeps tilting down \( \backslash \) do you know\( \backslash \) down down, as that point get closer and closer am as soon as had a point hit that point \( S \)-slams into \( T \). Becomes the same thing. That was the intuition I gave for how do you define the limit. How did you define of derivative. let me write that down again. But does everyone gets the geometric intuition. Take the point up there and slide it down. As you he slide it down you still have the secants line. It keeps twisting and turn and finally lands on top of this guy. that's just the simple picture. So in words, as you slide, give it a name, so I can write it down. Let me call this point \( P \)-for the sake of that -- call at that point \( Q \). Have name there. As you slide \( Q \)-towards \( P \), along the curve, so in algebra terms, as \( H \)-get smaller, the line \( S \), the secant line, gets closer to the tangent line. Just sort of inat this tiewf1. Two lines are getting closer and closer together. Their slopes are getting closer and closer.
together. Intuitively. And so and what's the slope of S? Well I wrote that down. It's just how much goes up divided by how much goes sideways. That gets closer to what's the slope of T? Well that's the derivative. That's the thing I want. And the way we wrote that down last time, let H-get arbitrarily small. That's the process. And the way I wrote it down is figure out somehow what it means, the way we wrote this down is I want H- to be arbitrarily small. So what I want to do is take the limit, the limit as H-gets as close as possible to zero. And the notation as H- goes to zero of this quantity are of that slope of secant line. So there was the definition of derivative I wrote down last time. This means slide those two points until they're just about on top of one another. And I just sort of appealed to your intuition to say la this notation means. Keep adding H- until it get smaller and smaller. So what I want to do now is talk about what limit of U-means. What this limit as H- goes to zero means. So now that brings up section 1.4, limit and derivatives. I want to talk about what limit means in general, we'll mostly use it for derivatives but I want to talk about it in general. So here is the formal definition of limit. Okay. Let G-of X-be a function. And A-is some number. So we say in English, L is the limit of G-of X-as X-approaches the number A-and in math notation we write the limit as X-approaches A-of G-of X-equals L, that's this is the math notation, that's the English. That's the way you would read it. The limit of the function G-of X-as X-gets closer to A-equals L if G-of X-can be made as close as you like, arbitrarily close to L by making X-arbitrarily close to A-but not equal to A. That's a little bit leery but that's part of the definition. So that's the formal definition of this limit process. We'll use it for derivative but it's the general definition of what it means to have a limb. limit. Do some examples to show you what this mean much simple functions. So let me take the function G-of X-equals X-squared. So it's very familiar parabola. And I'm going do let A-be .1. So take the limit as X-approaches one of X-squared. And so let me draw the points here. One comma one am so what happens to this function as X-gets closer and closer to one? You look at it and it's pretty obvious. If X-is really close to one then X-squared is really close to one. So I don't want to be anymore formal than that. It's natural. So let me do a few more examples to stretch your intuition and what it mean. So let me define G-of X-to be this. X-if X-is less than one, it's one is (on board). Okay. That's the function,
what's it look like? Here's .1. Over here, function X. Over here it is the function one but what happens at one? Draw a circle there but that's not the value. It's right there. That's a perfectly legitimate function. For every value of X-I tell you what the function is. Before you it jumps there. I'm allowed to write eye function like that. Again I'm going to ask what is the limit as X-goes to one of G-of X. There's definition. And now I'm asking you to pay attention to the whole definition but especially the last strays in (thesis and ask what did it mean. Ask yourself what it I get closer from this side but I don't actually go there. I don't let met self get equal to one. If I -- I'm not allowed to actually be there. Doesn't matter what happens at that point. And so the answer is still one. What happens at this point is irrelevant. I don't care what happens when X-is equal to A-but just had a happens really close on the left and really close on the right. That's the definition of limit. So let me do it one more time. Just to hammer it. Here's the function. Almost the same function. Just left out the middle part, what happens at zero am so let me draw the function again am there's .1. Looks like this. There's a hole there. Go off like that. Now what's with the function G-at zero? It's knots even defined. It's not even named. I haven't defined the function value yet. So note, G-of X-is not defined at X-equals one. The language we used earlier in the semester was one is not in the domain. That is the set where it's defined, of G. Okay. I'm allowed to write a function like that. Square root, used to that not being did he find. There's in function. What is the limit? Again same question. The limit as X-goes do one G-of X, so let me pull this down to we can read it. Again it's this phrase here what is most important am I don't care what happens at one am didn't have to be defined am still one. Close from this side it's getting closer to one. It doesn't matter what happens here am so that's what limits. So any questions? So what's more important. Put both fraidzs near the top. phrases near the top. So let me do some examples where limit don't exist just to see what that means.

So here's some examples where the limit does not exist. So you can write down this, write down this notation here but there is no L for which this is true. So draw some examples. How about G-of X-equals X-if X-is less than or equal to one. And that's what we were doing there. But now two X-is X-is greater than one. So let me draw that function. So it looks like this. Straight line over there. But if I'm just a little bit bigger, it looks like this. Goes up to the slope of two. And on this side is the function two X. And this side is the function X. So this is the .1 comma one and that circle that empty hole there put one comma two. That's what this function looks like. If defined everywhere. And so let me ask what could this possibly equal? I'll do the same thing as I did before. What happens if I approach from the left. Get closer to one. What happens if I approach from the right? I'm getting closer to two, I can't get closer to one and two at the same time so this limit does not exist. It's not gettting close to an single number when you're close to one. On this side you're close to two and that side you're close to one. So it's not
getting close to a single number so $G$-of $X$-does not get close to any single number $L$ when $X$-is close to one. That's what you need to have a limit. It has to be close to one number. It's close to two numbers. Depending on which side our on. Let me do one more example. How about $F$-of $X$-equals one over $X^2$ squared. What's this look like? When $X$-is tiny the function gets larger and larger. It blows up. So here, I can ask what's the lymph $G$-of $X$-as $X$-now approaches distance if it doesn't get close to any number. It keeps getting bigger and bigger. It also does not exist because in this case $G$-of $X$ keeps getting bigger. It does not get close to any particular number $L$. Just keeps getting bigger am so there's no limit. So.

STUDENT: You can say I the limit is infinity.

PROFESSOR: So people would say the number is infinite, capital $L$ is infinite. But I'm meaning this to be a finite number. So formally it would not exist. Co local we'll will I. So what if I did the function $G$-of $X$-equals one over $X$, who's plot this hyperbola. And on this side it goes to minus infients on this side it goes to plus infinite. So we don't want it talk about too much. There's two.

Let me do one more interesting example. Where there's no limit. So here there was no limit because the function was sort of like different on the left and right. Okay. Here there's no limit, the function is the same on the left and right but it goes to infinity. Can you think of a function $G$-of $X$-where it's bounded, so this doesn't go wrong. That's not a problem. It's always a nice finite number. And $G$-of $X$, I'm going to be interested in taking the limit as $X$-goes to zero so I want it to be the same on the left and right. Exactly the same function on the left and right. So it look the same when that side is that side. But still there's no limit. So there's still no limit. Still doesn't approach any number even though it's the same on both sides and not allowed to fly away to infinity. So just to have one more interesting example.

So let me sort of describe how you build such a function. There are a lot of ways to do it. And let me just do it this way. So here's zero. Here's horizontal line. And let me put a bunch of points in there. So there's points one. There's points one half. There's points a 30. A quarter am all those points. Draw a function now. Just going do draw the curve. So let's start here. And I will go up to one there. Just a straight line. And two and a half and third I go back down do zone am between a third and quarted I go and up and then down. That's what it looks like. I just keep defining much however close you gets you go up and down between the next two. One over $N$-and and one over $N$-plus one. So there's no limit here. here. Because however close to zero you get, and I would make the function look exactly the same on this side. So it goes up, down, up and do you think?
A \ do you know \ down. Look the same on both sides. So no limit exists because what number does G-get close to when you get close to zero? Because G-of X-get close to all numbers. Between zero and one. As X-goes to zero. Just keeps bouncing up and down faster and faster.

STUDENT: I have a question on G-of X-equals one over X-squared am couldn't you prove it is infinity because you could put in. Ohio Ohio.

PROFESSOR: It does go into infinite because in it class we're only talking about finite real numbers. It's hard to find out arithmetic with, what does infinity minus infinite mean. It's not a number. I can't define it. So lever as a special case. It's going, it gets as big as you like and you can write that down by saying the limit equals infinity. But that's shorthand so local we'll way it say it gets as big as you like. It didn't approach my single finite number. All our numbers are finite. So if you want it try to plot this function on your calculator, this is how you do it. Just plot this function. I didn't cosine one of X. That will do it. It will be smooth. So you can plotting that near zero. It will do the same thing.

Okay. So what we want now are rules to Royce when these exist. And so that you can say I see this function I understand it, it's easy it write do you think the limit. These examples are plent to show you what the limit of the definition are. So you know you can be a safe driver. Most of time it work pretty easily and I'm going to give you rules on how to use it.

So here we have, just give you a bunch of rules, called limit theorem or if you like, rules to make doing limits easy. And we'll use these rules to compute derivatives but the rules exist just sort of in general. So let's suppose I have two simpler functions. And I know their limit. So suppose I know the limit of, that X-goes to A-of F-and I know this one, the limit as X-goes to A-of G-of X-. Suppose I know them both. They're okay. They exist. So what I want to have is simple rules for new functions at that depends on F-and G. So the simplest thing you can imagine, that's the simplest thing you can imagine. The rule says you can factor out the, pull the constant K-out of the limit and it just pulls through. So that's easy. So let's just do the limit as X-approaches one of three X, this says, that's the limit of three times the limit as X-approaches one of X-and I think you can all, the limit of X-as X-approach one is obviously one. And so you can just factor out constant. So that makes liemp life easy. easy. The second rule says if you take powers, now taking a function to some positive power. I don't want to worry about dividing by zero. So take positive powers am squares or cube rootsz. That sort of thing. Take out the power it says. Take the limit, do that first and then take the power after that. you can just do that. and the simple example again, let me do a example here. So the limit as X-approaches one of X-squared, that's the
example I did before, that says take the limit first, and square it. And I get the obvious thing is one squared is one. So no surprises.

So the third rule is actually several rules all combined together. Let me do it this way. The limb as X-goes it A, F-of X-plus G-of X. Is not, you just take the limit the part separately and add. I can say this is the limit of sum equals the sum of the limits, that's an easy way to remember it. But I can also not just do addition. I can also do subtraction, it's the same idea. Limit of the difference is the difference of the limit. And I can momentum them. multiply them. So the limit of the product is the product of the limits so. Those are all threes rules together. That's not so hard am take the limit separately and add, subtract, multiply or whatever you're supposed to do. And all right, I think I did it already. Two fact I had over there. I know the limit of three X-is. In the limit of X-squared. So I just solve those problems. And that was three pus one equals four am so so as far so good. Now I'm missing one of the four base operations. What could go wrong? Why deny I have N. So if the limit of G-of X-is not zero then I can go ahead and divide by it. That's, so the limit of F-of X-divided by G-of X-is as X-goes to A-is just the quotient of the limits. But that does not help me with this because the numerator, denominator's going to zero and the numerator is going to zero, so zero to zero doesn't help. So I have to be a little bit more clever for calculus. So let me just reiterate that. Do we need this limit to be non zero for the limit to exist and all the, and the answer is no, because of that derivative let me just give you a simple example. So what if A-equals zero. And F-of X-equals X-squared and G-of X-equals, let's say F-of X-equals three X-and G-of X-equals X. So the limit as X-goes to zero of F-of X-divided by G-of X-is the limit as X-goes to zero of three X-divided by X-. That's not ray big deal. The X-s cancel. And I'm using the fact here remember that the limit X-does not equal zero. It only get arbitrarily close. That's not zero and that's no zero and I am allowed to cancel them. So this limit is perfectly happy, but could I do this? Board bore. (on board). If I take the limit and then divide I get zero divide I by zero. And it all goes wrong so I have to be careful. And every derivative example I do will be like that. I can't just take the limit of the top I get zero, I get zero. Let's do examples now of all these rules.

So try to leave all the rules uncovered. Is that good enough? And this one I'm going to put up.

STUDENT: Question. So if you come across the problem where you can't cancel out, the numerator and the did he norm naught.

PROFESSOR: Tbu the denominator is still going to be zero just you wait. I have a good example for that. That's an excellent question. So the answer is we're going to have to do some trig algebra to if I had he will with it so it will cancel. And I'm going to show you how. It's always good whether the students introduce the next topic. Let's start with an easy one.
So what is the limit numerator. Four squared (on board). So I, this is zero over zero. So let's see if I can fix that. So can I factor this? Anybody remember how to factor that polynomial. X plus four times X-minus four divided by X-minus four. Okay. So I'm asking you to remember that that product of those two polynomials is that one. So now am I allowed to divide those two. That's what I want to do is cancel them. Item I allowed to do that. Because if the definition of limit X-is not of exactly equal to four am so that's not zero and that's not zero. I'm allowed to cancel, it's perfectly okay. And I get X-plus four. So now I inside it use a rule. This is easy but let's use the rule. Now I inside to use a rule. This is easy. So tell me what rule I just used? It's rule three. The limit of the sum is the sum of the limits. So I just used that rule. Which I have up there. And now this is obviously four. Limit of X-as X-goes to four am that's four and the whole thing equals eight. So that's one example.

Let me do a few more. How about, so are there any questions about the algebra.

STUDENT: Why did you have to use the their rule. Why don't you plug in four.

PROFESSOR: That would be a perfectly good answer. answer. I was trying to go step-by-step. But feel free is to skip at that one because it was easy. Limit as X--- (on board). What happens if I plug in X-equals zero to the denominator. Zero. X-equals zero up here, zero. So what I'm going to do is another standard trick. Here I divided out a factor of one. This effect was one I made it go away am here I'm going to introduce a factor of one. Let me erase this. Right this is eight here. Here's what I'm going to do. I'm going it take this and multiply it by a factor that's equal to one. Numerator's the same as the denominator am it's one. When X-is tiny, this isn't anything, never close to zero. That's never close to zero. So why did I do that? Let's see what happens when I multiply it out. So maybe to see the pattern, let me, so this is something minus two, times something plus two. Who, just think of this, as some number variable. variable. Call it Y. So it's Y-minus two times Y-plus two. And what's that? It is this squared minus that squared. Right? So let me just, so Y-minus two times Y-plus two equals Y-squared minus four for Y-equals this contraption there. X-plus four. So just going to use that factor. So I get the square root of X-plus four squared am that's good, make the square root go away am minus two squared divide I by everything else. Okay. He shall buy the algebra so far? So now what happens, now, something nice will happen. So what happens when I take the joot squared, it all goes away. So the numerator look like X-plus four minus four. Okay. Nothing happens. And the denominator is X-times this thing. Plus four minus four. goes away. So now the numerator is X-has a factor X. The denominator as a a factor X-. I can cancel them too. So that goes away, and become one.
A And that goes away. So let me erase this little bit here. Doing the algebra. So I get the limit as X goes to zero of one divided by the square root of X plus four plus two. Okay. So I've just done algebra and now what happens when you plug in X equals zero into the numerator am it's one, it's constant. What happens to the denominator when X is close do zero. Plug if zero there and get zero plus four, square root of that is two. So I get the limit, so I get one divided by the square root of four plus two, that's one over two plus two and the answer is one quarter.

STUDENT: You said you plugged in zero but I thought you can't assume that.

PROFESSOR: If the numerator. What rule did I just use? I jut used a rule there without naming it. It's rule four this is rule four I just used. The denominator if I plug if zero it's not zero.

STUDENT: You said you can't ever assume X equals zero.

PROFESSOR: So now I'm saying this is a nice smooth, this function, nothing bad happens here at X equals zero. So, if I filled in more steps. If you were to fill in more steps. Let me try it verbally first. As X gets close to zero X plus four gets close to four. The square root of X plus four gets close to the square root of four which is two. So this plus that gets close to two plus two. So you've used actually all the rules but I didn't write them down. So when I said take the square root, that's okay he was using the power rule. Should I write this. When I said X plus four, that limit's okay. I was using rule three. It's okay to add. I think what the limit of X it is as X goes to two. And when I sum them I just have to sum limit. Rule thee tells me I can just add. Zero, rule two tells me I'm allowed to take the square root. It's okay to take the square root and I get the square root of four and I get two. And then I use rule three again that says it's okay to add. This is has a limit, that has a limit, go ahead and add the limit and I get two plus two and it's four am so the limit of denominator is four. And so this rules tells me I'm allowed to go ahead and divide. I didn't fill in every step on the board but I just filled in verbally. You're allowed to skip those kinds of steps.

A This is okay, this is a nice function of nothing goes wrong. All these rules apply and if I plug in X equals zero, it's not distortion I can go ahead and divide by it. Is that okay. Any other questions.

STUDENT: So you're saying if you would ever plugged in minus two instead of plus two --

PROFESSOR: Then, that's what I started with here and I got zero.

STUDENT: I'm talking about your denominator.
PROFESSOR: If I had a zero in the denominator I would still, I won't be done yet. So, well, so you're saying.

STUDENT: Change that symbol instead ever making it X-plus make it the square root of X---

PROFESSOR: Let me write it down, say what is that.

STUDENT: If you plug in zero now you get zero.

PROFESSOR: Yes and here the limit does not exist because this is one divided by something that gets very tiny. And if you were to plot it it looks like this. Something like that. Blowing up. Because it's one divided by a tiny number. This limit does not exist. Because the graph of the curve looks like it's blowing up. Something like that. Looks like that. I think you get something like a hyperbola. So that limit does not exist. I got to a point where the denominator was not zero and is done. Because I'm allowed to divide by something that's not zero that's okay. rule four told me I'm allowed to divides by non zeros.

STUDENT: So can we just derive the numerator and denominator.

PROFESSOR: If they're both non zero, then this says, yeah, just go the limit separately and go ahead and divide.

STUDENT: Derived. If the limit. If the numerator and denominator are both zeros can we just did he riefer the numerator and denominator separately and plug in zero.

PROFESSOR: And then he get zero divided by zero and you're not done because you don't know what that is.

STUDENT: In this problem I just took the derivative of the numerator and then I took --

PROFESSOR: So you're using a fancy idea call Lopital's rule. So that's something that would be in imagine, that's way later in the semester. What you're saying is true, I didn't want to cover it now.

STUDENT: I'm sorry --

PROFESSOR: Are you allowed to use a fancy idea from another course. Let's talk about it off line okay.

STUDENT: Okay.
PROFESSOR: Okay so let me, that was a good example. Got lots of discussion going. That's fine. Let me do now another example which is really a derivative. Where we get zero to zero because that's the way we set it up.

Okay. So define a function. And what I want to compute is the derivative of that function. Okay. I want to use all these rules, compute the derivative. So here it is. Here's the definition. (on board). Of a derivative. So let me just plug it in and make it all work. Okay so this is the function says add three and take the square root. So add three to the argument X-plus H-and trick the square root. So I just did substitution. So this is setup. It is the derivative. If I plug in H-equals zero I get zero divided by zero. So I have to do something more clever. And I'm going to use exactly the same trick that I just erased. I'm going to multiply the top and bottom but this contraption with the plus sign.

STUDENT: On the difference quo shernt shouldn't that be.

PROFESSOR: You are absolutely right. Okay. (make correction). So it's exactly the same trick as I just used. So there's the same thing as before. And momentum momentum it by one. multiply it by one. Complicated way to write one. (on board). Momentum rate and denominator are the same and when you plug in H-equals zero nothing goes wrong am numerator and denominator are nice positive big numbers. If I multiply these two together it's the same trick, I get that guy squared minus that guy squared that is the algebra. So I get the square root of X-plus H-plus three squared minus the square root of X-plus three squared. And then I get H-and all this business. (on board). So now the square root and the squares cancel one another. So I get X-plus H-plus three minus X-minus three divided by H-times all this stuff. Okay. It's setup so that the X-'s cancel. The three's cancel. As that's live of left is H-guided by H-. They cancel. I'm left with one up in the null rate. So the limit of H, board bod. And we're back to where we had that long discussion before, it's perfectly all right it take the limit of the top. That is one. And the limit of the bofort just plug in H-equals zero, nothing dispose wrong. The denominator is non zero. So it's one divided by, what happens when I plug in H-equals zero I get the square root of X-plus three plus the square root of X-plus three, so I get two of them and there they are. And there is the formula for the derivative. For which we'll have a rule eventually. So I did it here by lands so you know you can do it in principle by going back to the basics, but we'll have a rule for these that I'll write down later. But now you know. Believe it. It's all the way down to the basics.

So last time remember I told you something will the power rule for derivatives. Just going to remind you we talked about this last time. If R- is any old number, suppose we remember that rule and now I ask you to compute this limit. Can you do it without working very hard? So let's just, remember that rule, and I want us to recognize, suppose we have to
do this limit. If you plug in X-equals zero you get zero divided by zero so you can go at that way. Do you recognize, I'm just going to compute a derivative of some function using that power rule. So I can say I don't have to do this limit this complicated way. I'll just apply power rule. This trick here worked with the square root. It doesn't work so well with (inaudible). So let's not do that. let's use the affect that we have this rule here am I know how to do that. So what function do you think I'm applying, I'm differentiating. That sort of mampedz that. Taking something to the third power. Let me write it this way. The limit as H-goes to zero (on board). That's the same thing, I just changed X-it H, it's the same limit and changed one to cubed root of one am so now do you recognize what I'm doing? I'm taking the derivative of some function, right? Which function? So let's just take, what's the derivative of X-to the one third? It's one third X-to the one third minus one which is one over three X-to the two thirds. Okay. so, and but the other thing as we know this is also the limit as H-goes to zero of X-(on board). Divided by H. That's the other anything. -- this pattern matches that pattern. But I know what this limit is because I have this power rule. So what is this really? Without doing any work? What I'm saying is this is really the derivative, this is the derivative of some familiar function, evaluated at some point, at some value of X. So what if I, let me change the one to an X. And say X-equals one. That is the same thing. Just changed one to X. Same thing. This is now that. This is the exactly the derivative. So this is exactly equal to one over three times X-to the two-third at X-equals one. And so it's one third. So the point here was that if you know the power rule which I talked about last time and reminded you what it was. And said if you want it differentiate X-it any power, put -- and subtract one from it. Here the point was I wrote down this funny limit and the idea was, I'm really just differentiating this. The problem didn't say differentiate X-to the one third but I recognize it's the same problem, it's the same limit. So I don't have to do all at that algebra in order to figure out what the limit is, I can just recognize that I've solved the problem already am differentiating the function X-to the one third power. And plugging in X-equals one. That's sort of a pattern recognition thing that's a useful skill too. So I think that one deserves a homework question to practice. So I will ask one.

Let me say one tiny thing about limits to finish this section. Equal L if the value of F-of X-gets closer to I can't plug in infinity. The numerator get larger and larger and the denominator get larger and larger am I have to do something else it make this work. Let me take both numerator and denominator and divide by X. So I take this thing and multiply that by one over X-and that by one over X-. Multiply the whole thing by one and it look like this. So what happens to the numerator as X gets larger and larger am it approach six. And what happens to the denominator as X-get larger and larger. It approaches two. So I can use my rule, whatever it was, rule four, and I can get six divided by two or three. So I can use the rules there too. So that is the ends of section 1.4. All about limits.
We're not done with limit but I'm done defining them and making sure we understand what they mean. So ...

Let's go on. Section 1.5. So what this section, so the last section was about how do you look at a limit and decide whether it exist and what value. So now this section has to sort of ask the same question which is how do we tell the function that somebody hands us is differentiable at X. In other words, if F-prime of X-exists or if you like how did you tell if this limit exists? Without going through the whole definition. So is there in easier way to do it. You can't just plug many H-equal zero. The numerator is zero and denominator is zero. So something goes wrong. But here is a simple observation, give you something has to be true of F-for this to have a hope of existing. Since the noam narrate is going to zero what does the numerator have to go to? It also has to go to zero since H-dispose to zero, the numerator has to go to zero. Must also go to zero. So otherwise the quotient is obviously going to go to infinity. So the numerator has to go to zero. So let me write that down. That means that the limit as H-approaches zero of F-of X-plus H-minus F-of X, the limit -- limit as H-goes to zero of F-of X-plus H-minus the limit as H-goes to zero of F-of X, that has to be equal to zero. Or just rearrange it slightly one more time, the limit as H-goes to zero of F-of X-plus H-equals the limit (on board), which is constant so that's just F-much X. Simply says that as the argument as the function gets closer to F-of X. And this is so important, it gets its own name. Definition, function of F-of X-is continuous at a particular point X-if this limit holds S. so that's an incredible simple property of a function. And most of the functions we've written down on the board have that property. we draw that function already. We draw at that, pick up your pencil. To get across. If you can't do it. And we also had this example. So I had one. A half. A third. A quarter. quarter. And you draw it like this. It kept bouncing. And I guess I kept my chalk on the paper. My pence he will on the paper. My chalk on the board. Graph the function without lifting your pencil often the paper or running out of lead. Obviously you'd run out of lead before you got to zero. So that's, so none, all of these are discontinuous. You don't get close to a single value as you get close to zero. So this is a simple idea. And the question is is that all we need for a function to be differentiable? If the function, if I can draw the function without lifting my pencil often the paper is that good enough for it to have a derivative? Here's the question. (on board). Okay. So I can draw the function. Without lifting my pencil offer the board am can I always find a tangent line at every point? Let me do a little example. How about the function F-of X-equals the absolute value of X? Let's me just graph that. So on this side, I'll write down the definition for fun. So it's X-if X-is gearnlts greater than or equal to zero. That's a nice continuous function, I draw it without listtion my chalk off the board, no problem. So over here it's a straight line. It has a derivative. It's slope is one. Over here, it's a straight line, it has a derivative. Its slope is minus one. So here the slope equals the derivative equals plus one. On this
side. Over here, the slope equals the derivative equals minus one if X is less than zero. So the question is what happens there? Can you draw a tangent line there? So there's a candidate. And there's a candidate. There's a candidate. Right for a tangent line. So the question is does F-prime at zero exist? So does it a derivative there? There a taken line. So the question is does the limit as H-goes to zero of (on board), I put this down many times, does this exist at X-equals zero, so let me keep doing that algebra. So does that exist? So let me write it down and see. So limit as (on board) divided by H-. That is what it is when you plug it all in. (on board). So let's plot that function. So I want to plot absolute value of H-divide by H. When H-is positive it's H-over H. Whrks H-is negative, it's minus H-over H. Is that function have a limit? As H-gets closer to zero? No limit. Does not exist. So this is not differentiable at X-equals zero. So continuity is not an continuity alone does not guarantee different shaibility.

STUDENT: It work the other way though right.

PROFESSOR: Yes. The other way it work. Good observation. If it's differentiable, then what I proved over here is that it has to be continuous. The numerator has to go to zero but it doesn't work the other way. And so any. So let met just say these kinds of functions are not uncommon.