3.1.18) Differentiate \( y = \frac{7}{9} + \frac{x^2 + x + 1}{x^2 + 1} \)
\[
y' = \frac{(x^2+1)(2x+1)- (x^2 + x + 1)(5x^4)}{(x^2+1)^2}
\]

3.1.26) Differentiate \( y = \frac{x+2}{(x+3)(x+4)} \)
\[
y' = \frac{(x+3)(x+4)(1) - (x+2)((x+3)(1)+(1)(x+4))}{(x+3)^2(x+4)^2} = \frac{x^2 + 7x + 12 - (2x^2 + 11x + 14)}{(x+3)^2(x+4)^2} = \frac{-x^2 - 4x - 2}{(x+3)^2(x+4)^2}
\]

3.1.34) \( y = (x^2 - 1)^4(x^2 + 1)^5 \) Find the coordinates of the local maxima and minima.
First, find the derivative: \( y' = (x^2 - 1)^4 \cdot 5(x^2 + 1)^4(2x) + (x^2 + 1)^5 \cdot 4(x^2 - 1)^3(2x) \), so
\[
y' = 2x(x^2 - 1)^3(x^2 + 1)^4[5(x^2 - 1) + 4(x^2 + 1)] = 2x(x^2 - 1)^3(x^2 + 1)^4(9x^2 - 1).
\]
Clearly, \( y' = 0 \) for \( x = 0, \pm 1, \pm 1/3 \), so these are the critical points. It is clear from the graph that 0 and \( \pm 1 \) give local minima, and \( \pm 1/3 \) give local maxima. Plugging these values into the equation for \( y \), we get local minima at \((-1,0),(0,1),\) and \((1,0)\) and local maxima at \((\pm 1/3, \pm 10^5)\), or \( \approx (\pm 0.33, 1005) \).

3.1.44) \( h(x) = (\frac{f(x)}{x})^2 \) Find \( h'(x) \).
\[
h'(x) = 2(\frac{f(x)}{x})^1 \cdot \frac{x f'(x) - f(x)1}{x^2} = \frac{2xf(x)f'(x) - 2f(x)^2}{x^2}
\]

3.1.50) Let \( s(t) \) be the number of miles a car travels in \( t \) hours. Then the average velocity during the first \( t \) hours is \( \bar{v}(t) = s(t)/t \) miles per hours. Suppose the average velocity is maximized at time \( t_0 \). Show that at this time the average velocity \( \bar{v}(t_0) \) equals the instantaneous velocity \( s'(t_0) \).
\[
\bar{v}'(t) = \frac{t \bar{v}'(t)- \bar{v}(t)1}{t^2} \quad \text{and this is true for all } t, \quad \text{so } \bar{v}'(t_0) = \frac{t_0s'(t_0) - s(t_0)}{t_0^2}.
\]
Since \( \bar{v} \) is maximized at \( t_0 \), \( \bar{v}'(t_0) = 0 \), so \( t_0s'(t_0) - s(t_0) = 0 \). Thus, \( t_0s'(t_0) = s(t_0) \), \( t_0 \) s'(t_0) = s(t_0), so \( s'(t_0) = \frac{s(t_0)}{t_0} = \bar{v}(t_0) \), which is what we are trying to show.

3.1.56) Find the coordinates of the minimum point of \( y = \frac{1}{2} + \frac{x^2 - 2x + 1}{x^2 - 2x + 2} \) for \( 0 \leq x \leq 2 \).
\[
y' = \frac{(x^2 - 2x + 2)(2x - 2) - (x^2 - 2x + 1)(2x - 2)}{(x^2 - 2x + 2)^2} = \frac{(2x - 2)(x^2 - 2x + 2 - 2x - 1)}{(x^2 - 2x + 2)^2} = \frac{2x - 2}{(x^2 - 2x + 2)^2}.
\]
This has critical points at 1 and when \( x^2 - 2x + 2 = 0 \), which only happens in the complex numbers. Thus, the only critical point we need to consider is \( x = 1 \). Plug in values on either side to show 1 is a minimum. \( y'(0) = -2/4 = -1/2 < 0 \), \( y'(2) = 2/4 = 1/2 > 0 \), so 1 is a minimum. Thus, the coordinate of the minimum is \((1/2, 1)\).

3.2.4) \( f(x) = \frac{x+1}{x-3}, \) \( g(x) = x + 3 \). Compute \( f(g(x)) \).
\[
f(g(x)) = \frac{x+3+1}{x+3-3} = \frac{x+4}{x}
\]

3.2.10) \( h(x) = (4x - 3)^3 + \frac{1}{4x-3} \) Find \( f(x) \) and \( g(x) \) such that \( h(x) = f(g(x)) \).
\[
f(x) = x^3 + \frac{1}{x}, \quad g(x) = 4x - 3
\]

3.2.16) Differentiate \( y = 2(2x - 1)^{5/4}(2x + 1)^{3/4} \)
\[
y' = 2(2x - 1)^{5/4} \cdot \frac{3}{4}(2x + 1)^{-1/4}(2) + 2(2x + 1)^{3/4} \cdot \frac{5}{4}(2x - 1)^{1/4}(2)
\]
\[ y' = (2x - 1)^{1/4}(2x + 1)^{-1/4}(3(2x - 1) + 5(2x + 1)) = (2x - 1)^{1/4}(2x + 1)^{-1/4}(16x + 2) \]

3.2.26) \( h(x) = \sqrt{f(x^2)} \). Find \( h'(x) \)

\[ h'(x) = (1/2)(f(x^2))^{-1/2}f'(x^2)(2x) = x(f(x^2))^{-1/2}f'(x^2) \]

3.2.28) Sketch the graph of \( y = 2/(1 + x^2) = 2(1 + x^2)^{-1} \).

\[ y' = 2(-1)(1 + x^2)^{-2}(2x) = -4x/(1 + x^2)^2, \) so \( y \) has a critical point at \( x = 0 \). It is clear that \( y' \) is positive to the left of 0 and negative to the right, so \( y \) in increasing on \((-\infty, 0)\) and decreasing on \((0, \infty)\) and has a local maximum at 0 with a value of 2. This is also a global max, since \( y \) is only increasing to the left and only decreasing to the right.

\[ y'' = (1+x^2)^2(-4)-(4x)(2)(1+x^2)(2x) = (1+x^2)^{-2}(4x^2+16x^2) = -4x^2(1+x^2)^{-2}. \) This is 0 when \( 12x^2 = 4 \), i.e. when \( x = \pm \sqrt{1/3} \). We see that for \( x = 0 \) \( y'' \) is negative, but it is positive for \( x = \pm 1 \), so \( x = \pm \sqrt{1/3} \) are points of inflection, and \( y \) is concave up on \((-\infty, -\sqrt{1/3}), (\sqrt{1/3}, \infty) \) and concave down on \((-\sqrt{1/3}, \sqrt{1/3}) \).

The \( y \)-intercept is (0, 2), and \( y \) has no \( x \)-intercepts, since it can never equal 0. It is defined everywhere, and has asymptotes at \( y = 0 \), since \( \lim_{x \to \infty} y = 0 \) and \( \lim_{x \to -\infty} y = 0 \)

3.2.34) \( f(x) = \frac{4}{x} + x^2, g(x) = 1-x^4 \). Find \( \frac{d}{dx} f(g(x)) \).

\[ \frac{d}{dx} f(g(x)) = \frac{d}{dx} \left( \frac{4}{x} + (1-x^4)^2 \right) = \frac{-4x}{x^2} + 2(1-x^4)(-4x^3) = \frac{16x^3}{(1-x^4)^2} - (1-x^4)(8x^3) \]

3.2.40) \( y = \frac{u^2+2u}{u+1}, u = x+1 \). Find \( \frac{dy}{dx} \).

\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{(u+1)(2u+2)-(u^2+2u)(1)}{(u+1)^2} \times [x(1)+(1)(x+1)] = \frac{2u^2+4u+2-u^2-2u}{(u+1)^2}(2x+1) = \frac{u^2+2u+2}{(u+1)^2} (2x+1) = \frac{2x+1}{(2x+1)(x^2+1)} \]

3.2.52) Suppose \( Q, x, \) and \( y \) are variables, where \( Q \) is a function of \( x \) and \( x \) is a function of \( y \). a.) Write the derivative symbols for:

the rate of change of \( x \) with respect to \( y \) : \( \frac{dx}{dy} \)
the rate of change of \( Q \) with respect to \( y \) : \( \frac{dQ}{dy} \)
the rate of change of \( Q \) with respect to \( x \) : \( \frac{dQ}{dx} \)
b.) Write the chain rule for \( \frac{dQ}{dy} \): \( \frac{dQ}{dy} = \frac{dQ}{dx} \frac{dx}{dy} \)

3.2.56) A manufacturer of microcomputers estimates that \( t \) months from now it will sell \( x \) thousand units of its main line of microcomputers per month, where \( x = .05t^2 + 2t + 5 \).

Because of economies of scale, the profit \( P \) from manufacturing and selling \( x \) thousand units is estimated to be \( P = .001x^2 + .1x -.25 \) million dollars. Calculate the rate at which the profit will be increasing 5 months from now.

We need \( \frac{dP}{dt}(5) \).

\[ \frac{dP}{dt} = \frac{dP}{dx} \frac{dx}{dt} = (.002x + .1)(1t + 2) = (.002(.05t^2 + 2t + 5) + .)(1t + 2) = (.0001t^2 + .004t + .1)(1t + 2) = .00001t^3 + .0006t^2 + .019t + .22 \]

Thus, we get \( \frac{dP}{dt}(5) = .33125 \), by plugging \( t = 5 \).

3.3.16) Find \( \frac{dy}{dx} \) if \( x^2 + 4xy + 4y = 1 \)
Taking \( \frac{dy}{dx} \) of both sides, we get \( 2x + 4x \frac{dy}{dx} + 4y + 4 \frac{dy}{dx} = 0 \). So \( \frac{dy}{dx}(4x + 4) = -2x - 4y \), so \( \frac{dy}{dx} = \frac{-2x - 4y}{4x + 4} \).

3.3.18) \( x^3y + xy^3 = 4 \)

Taking \( \frac{d}{dx} \) of both sides, we get \( x^3 \frac{dy}{dx} + 3x^2y + x(3y^2) \frac{dy}{dx} + (1)y^3 = 0 \). So \( \frac{dy}{dx}(x^3 + 3xy^2) = -3x^2y - y^3 \), so \( \frac{dy}{dx} = \frac{-3x^2y - y^3}{x^3 + 3xy^2} \).

3.3.28) The graph of \( x^4 + 2x^2y^2 + y^4 = 9x^2 - 9y^2 \) is a lemniscate.

a.) Find \( \frac{dy}{dx} \) by implicit differentiation

Taking \( \frac{d}{dx} \) of both sides, we get \( 4x^3 + 2x^2(2y) \frac{dy}{dx} + 4x(y^2) + 4y^3 \frac{dy}{dx} = 18x - 18y \frac{dy}{dx} \). So \( \frac{dy}{dx}(4x^2y + 4y^3 + 18y) = 18x - 4x^3 - 4xy^2 \), so \( \frac{dy}{dx} = \frac{18x - 4x^3 - 4xy^2}{4x^2y + 4y^3 + 18y} \).

b.) Find the slope of the tangent line to the lemniscate at \((\sqrt{5}, -1)\)

slope = \( \frac{dy}{dx} \bigg|_{(\sqrt{5}, -1)} = \frac{18\sqrt{5} - 4\sqrt{5} - 4\sqrt{5}(1)^2}{4\sqrt{5} - (-1) + 4(-1)^3 + 18(-1)} = \frac{18\sqrt{5} - 20\sqrt{5} - 4\sqrt{5}}{-20 - 4 - 18} = \frac{-6\sqrt{5}}{-42} = \frac{\sqrt{5}}{7} \)

3.3.30) Suppose that \( x \) and \( y \) represent the amounts of two basic inputs for a production process and \( 10x^{1/2}y^{1/2} = 600 \). Find \( \frac{dy}{dx} \) when \( x = 50, y = 72 \)

We have \( x^{1/2}y^{1/2} = 60 \), so \( x^{1/2}(1/2)y^{-1/2} \frac{dy}{dx} + (1/2)x^{-1/2}y^{1/2} = 0 \). Then \( \frac{dy}{dx} = \frac{-x^{-1/2}y^{1/2}}{x^{1/2}y^{1/2}} = \frac{-y}{x} \).

So when \( x = 50, y = 72 \) \( \frac{dy}{dx} = \frac{-72}{50} = \frac{-36}{25} \).

3.3.36) Determine \( \frac{dy}{dt} \) if \( x^2y^2 = 2y^3 + 1 \)

We take \( \frac{d}{dt} \) of both sides to get \( 2x \frac{dx}{dt}y^2 + x^2(2y) \frac{dy}{dt} = 6y^2 \frac{dy}{dt} \). So \( 2xy^2 \frac{dx}{dt} = \frac{dy}{dt}(6y^2 - 2x^2y) \), and so \( \frac{dy}{dt} = \frac{2xy^2 \frac{dx}{dt}}{6y^2 - 2x^2y} \).

3.3.46) An airplane flying 390 feet per second at an altitude of 5000 feet flew directly over an observer.

a.) Find an equation relating \( x \) and \( y \)

5000^2 + x^2 = y^2

b.) Find the value of \( x \) when \( y \) is 13,000

\( x = \sqrt{13000^2 - 5000^2} = \sqrt{169000000 - 25000000} = \sqrt{144000000} = 12000 \), where we take the positive square root because we want a positive distance
c.) How fast is the distance from the observer to the airplane changing at the time when the airplane is 13,000 feet for the observer? What is \( \frac{dy}{dt} \) when \( \frac{dx}{dt} = 390 \) and \( y = 13000 \)?

From a.) we get \( 2x \frac{dx}{dt} = 2y \frac{dy}{dt} \). Plugging in \( y = 13000, x = 12000 \) and \( \frac{dx}{dt} = 390 \), we get \( 9360000 = 26000 \frac{dy}{dt} \), so \( \frac{dy}{dt} = 360 \) feet per second.

Supplementary Exercises: 26) \( h(x) = g(f(x)) \) Determine \( h(1) \) and \( h'(1) \).

\( h(1) = g(f(1)) \). Since \( f(1) = 3 \) and \( g(3) = 1 \), \( h(1) = 1 \). By the chain rule, \( h'(x) = g'(f(x))f'(x) \), so \( h'(1) = g'(f(1))f'(1) \). From the graphs, we see \( f(1) = 3, f'(1) = 1/2 \), and \( g'(3) = -1/2 \), so \( h'(1) = -1/2 * 1/2 = -1/4 \).

42a.) Find \( \frac{dy}{dx} \) if \( x^3 + y^3 = 9xy \)
We get \( 3x^2 + 3y^2 \frac{dy}{dx} = 9x \frac{dy}{dx} + 9y \) by taking \( \frac{d}{dx} \) of both sides. So \( \frac{dy}{dx} (3y^2 - 9x) = 9y - 3x^2 \), and so \( \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} \).

b.) Find the slope of the curve at \((2, 4)\)

\[
\text{slope} = \left. \frac{dy}{dx} \right|_{(2, 4)} = \frac{9(4) - 3(2)}{3(4)^2 - 9(2)} = \frac{36 - 12}{48 - 18} = \frac{24}{30} = \frac{4}{5}
\]

50) An offshore oil well is leaking oil onto the ocean surface, forming a circular oil slick about .005 meter thick. If the radius of the slick is \( r \) meters, then the volume of the oil spilled is \( V = .005\pi r^2 \) cubic meters. Suppose that the oil is leaking at a constant rate of 20 cubic meters per hour, so that \( \frac{dV}{dt} = 20 \). Find the rate at which the radius of the oil slick is increasing, at a time when the radius is 50 meters.

\( V = .005\pi r^2 \) gives us that \( \frac{dV}{dt} = .005\pi 2r \frac{dr}{dt} \). Plugging in \( \frac{dV}{dt} = 20 \) and \( r = 50 \), we get \( 20 = .005\pi 100 \frac{dr}{dt} \), so \( \frac{dr}{dt} = 40/\pi \) meters per second.