• **4.1 #14** Write each expression in the form of $2^kx$ or $3^kx$, for a suitable constant $k$; $(3^{-x} \cdot 3^{x/5})^5, (16^{1/4} \cdot 16^{-3/4})^{3x}$

*Solution*

$$(3^{-x} \cdot 3^{x/5})^5 = 3^{-5x} \cdot 3^x = 3^{-4x}$$$$(16^{1/4} \cdot 16^{-3/4})^{3x} = (16^{1/4-3/4})^{3x} = (16^{-1/2})^{3x} = ((2^4)^{-1/2})^{3x} = 2^{4 \cdot -1/2 \cdot 3x} = 2^{-6x}$$

• **4.1 #30** Solve $(2 - 3x)5^x + 4 \cdot 5^x = 0$

*Solution* Factoring out $5^x$, we get

$$(2 - 3x + 4) \cdot 5^x = 0 \rightarrow (6 - 3x) \cdot 5^x = 0$$

Since $5^x$ is never zero, we can divide both sides by $5^x$. So $6 - 3x = 0$, hence $x = 2$.

• **4.1 #32** Solve the following equations; $2^x - \frac{1}{2^x} = 0$

*Solution* Multiplying $2^x$, we get $2^x \cdot 2^x - 1 = 0$. Using exponential law, we get $2^{2x} = 1$. We get $2x = 0$ since $2^{2x} = 2^0$. Therefore, $x = 0$

• **4.1 #42** Find the missing factor $5^{7x/2} - 5^{x/2} = \sqrt{5^x}(\quad)$

*Solution* Note that $\sqrt{5^x} = 5^{x/2}$. Also note that $5^{7x/2} = 5^{x/2+6x/2} = 5^{x/2} \cdot 5^{6x/2}$. So we have,

$$5^{7x/2} - 5^{x/2} = \sqrt{5^x}(5^{6x/2} - 1) = \sqrt{5^x}(5^{3x} - 1)$$

So the missing factor is $5^{3x} - 1$.  

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• **4.2 #4** Compute the given derivatives using formula (2), (5), (6)–(8).

  **Solution**
  (a) By (6), \( \frac{d}{dx}(2^x) \big|_{x=1/2} = m2^{1/2} \) where \( m = \frac{d}{dx}(2^x) \big|_{x=0} \). So the answer is \( \sqrt{2} \cdot m \equiv \sqrt{2} \cdot m \equiv .98 \). (In fact, \( m = \ln 2 \). So \( \frac{d}{dx}(2^x) \big|_{x=1/2} = \sqrt{2} \ln 2 \)).

  (b) By (6), \( \frac{d}{dx}(2^x) \big|_{x=2} = m2^2 \) where \( m = \frac{d}{dx}(2^x) \big|_{x=0} \). So the answer is \( 4 \cdot m \equiv 2.52 \). (In fact, \( m = \ln 2 \). So \( \frac{d}{dx}(2^x) \big|_{x=2} = 4 \ln 2 \)).

• **4.2 #16** Write each expression in the form of \( e^{kx} \).

  **Solution** \( \left(\frac{e^3}{x^2}\right)^x = (e^2)^x = e^{2x} \) and \( e^{4x+2} \cdot e^{x-2} = e^{4x+2+x-2} = e^{5x} \).

• **4.2 #28** Differentiate \( y = \frac{1}{1+e^x} \).

  **Solution** We take derivative of \( y = (1 + e^x)^{-1} \). By the chain rule, we get \( y' = (-1)(1 + e^x)^{-2} \cdot e^x = -\frac{e^x}{(1+e^x)^2} \).

• **4.2 #32** Differentiate \( y = \frac{e^x}{x^2} \).

  **Solution** By quotient rule, we get \( y' = \frac{e^x(x^2) - e^x(2x)}{x^4} = \frac{e^x x^2 - 2}{x^3} \). Or you can apply product rule to \( y = e^x x^{-2} \). Then we get the same answer \( y' = e^x x^{-2} + e^x \cdot (-2)x^{-3} \).

• **4.2 #36** Differentiate \( y = (xe^x - 1)^{-3} \).

  **Solution** By the general power rule,

  \[ y' = -3(xe^x - 1)^{-4} \frac{d}{dx}[xe^x - 1] \]

  By the product rule, we get

  \[ \frac{d}{dx}[xe^x] = e^x + xe^x \]

  So, \( y' = -3(xe^x - 1)^{-4}[e^x + xe^x] \).

• **4.2 #38** Show that the tangent line to the graph of \( y = e^x \) at the point \((a, e^a)\) is perpendicular to the tangent line to the graph of \( y = e^{-x} \) at the point \((a, e^{-a})\).

  **Solution** We compare the slopes of two tangent lines. Since \( (e^x)' = e^x \), the slope of the first tangent line is \( e^a \). Since \( (e^{-x})' = -e^{-x} \), the slope of the second tangent line is \( -e^{-a} \). Multiplying these two, we get \( e^a \cdot (-e^{-a}) = -1 \). So they are perpendicular to each other.
• 4.3 #8 Differentiate $f(x) = 2e^{\sqrt{x}}$.
  
  Solution By chain rule, we get $y' = 2e^{\sqrt{x}} \cdot (\sqrt{x})' = 2e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$

• 4.3 #10 Differentiate $g(x) = (e^{-2x} - 2x)^3$
  
  Solution By the general power rule,
  
  $\frac{d}{dx}(g(x)) = 3(e^{-2x} - 2x)^2 \frac{d}{dx}[e^{-2x} - 2x]$

  Since $\frac{d}{dx}[e^{-2x} - 2x] = -2e^{-2x} - 2$, $g'(x) = -6(e^{-2x} - 2x)^2(e^{-2x} + 1)$

• 4.3 #20 Differentiate $f(x) = \sqrt{e^{x/2} + 1}$
  
  Solution Note that $f(x) = (e^{x/2} + 1)^{1/2}$. By the general power rule, we get
  
  $f'(x) = \frac{1}{2}(e^{x/2} + 1)^{-1/2} \frac{d}{dx}[e^{x/2} + 1] = \frac{1}{2}(e^{x/2} + 1)^{-1/2}(\frac{1}{2}e^{x/2})$

  So, $f'(x) = \frac{1}{4}(e^{x/2} + 1)^{-1/2}e^{x/2}$

• 4.3 #22 Differentiate $f(x) = e^{ex}$
  
  Solution By chain rule, $f'(x) = e^x \cdot (e^x)' = e^x e^x = e^{x^2}$

• 4.3 #32 $f(x) = (2x - 5)e^{3x - 1}$
  
  Solution We find the critical points by solving $f'(x) = 0$. Taking the derivative, we get
  
  $f'(x) = 2e^{3x-1} + (2x-5)e^{3x-1} \cdot 3 = (6x-15+2)e^{3x-1} = (6x-13)e^{3x-1}$

  So $x = \frac{13}{6}$ is the only possible candidate for extreme point. Now we consider the second derivative.
  
  $f''(x) = 6e^{3x-1} + (6x-13)e^{3x-1} \cdot 3 = (18x-33)e^{3x-1}$

  Since $f''(\frac{13}{6}) > 0$, it has local minimum at $x = \frac{13}{6}$.

• 4.3 #40 Find $\frac{dy}{dx}$ if $y = e^{(1/10)e^{x/2}}$
  
  Solution Recall the chain rule; $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$ Applying the chain rule, we get
  
  $\frac{dy}{dx} = e^{(1/10)e^{x/2}} \cdot \frac{d}{dx}[(1/10)e^{x/2}]$
Since \( \frac{d}{dx}[(1/10)e^{x/2}] = \frac{1}{10} \cdot \frac{1}{2}e^{x/2} = \frac{1}{20}e^{x/2} \), we get
\[
\frac{dy}{dx} = e^{(1/10)e^{x/2}} \cdot \frac{1}{20}e^{x/2} = \frac{1}{20}e^{(1/10)e^{x/2}+x/2}
\]

- **4.3 #44** Determine all functions \( y = f(x) \) such that \( y' = 3y \) and \( f(0) = \frac{1}{2} \).

  *Solution* Since \( y = f(x) \) satisfies the differential equation \( y' = ky \) with \( k = 3 \), \( y = Ce^{3x} \) for some constant \( C \). Since \( f(0) = 1/2 \), we can determine the constant;

\[
f(0) = Ce^{3\cdot0} = Ce^0 = C \rightarrow C = \frac{1}{2}
\]

So \( f(x) = \frac{1}{2}e^{3x} \)

- **4.3 #46** Let \( f(x) \) be a function with the property that \( f'(x) = 1/x \) Let \( g(x) = f(e^x) \), and compute \( g'(x) \).

  *Solution* By chain rule, \( g'(x) = f'(e^x) \cdot (e^x)' \). Since \( f'(x) = 1/x \) and \( (e^x)' = e^x \), \( g'(x) = \frac{1}{e^x}e^x = 1 \).

- **4.3 #48** As \( h \) approaches 0, what value is approached by \( \frac{e^{2h}-1}{h} \)?

  *Solution* Let \( f(x) = e^{2x} \). By the definition of the derivative, we have

\[
\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{e^{2(x+h)} - e^{2x}}{h}
\]

Substituting 0 for \( x \), we get

\[
f'(0) = \lim_{h \to 0} \frac{e^{2h} - 1}{h}
\]

Since \( f'(x) = 2e^{2x} \), we have \( f'(0) = 2 \). So \( \frac{e^{2h}-1}{h} \) approaches to 2 as \( h \) approaches to 0.

- **4.4 #16** Simplify \( \ln(e^{-2}e^4) \)

  *Solution* \( \ln(e^{-2}e^4) = \ln e^{-2+4} = \ln e^2 = 2 \)

- **4.4 #32** Solve \( e^{\sqrt{x}} = \sqrt{e^x} \)

  *Solution* Squaring both sides, we get

\[
e^{2\sqrt{x}} = e^x
\]

So we get \( 2\sqrt{x} = x \). Squaring again, we get \( 4x = x^2 \). So we get \( x = 0, 4 \). One can check that \( x = 0, 4 \) are solution of \( e^{\sqrt{x}} = \sqrt{e^x} \).
• 4.4 #38 Solve \((e^2)^x \cdot e^{\ln 1} = 4\)

Solution

\[ e^{2x} \cdot 1 = 4 \quad \text{(since } e^{\ln 1} = 1) \]

\[ \ln (e^{2x}) = \ln 4 \]

\[ 2x = \ln 4 \]

\[ x = \frac{1}{2} \ln 4. \]

• 4.4 #40 Find the coordinates of the maximum and minimum points.

Solution \( f(x) = -1 + (x - 1)^2 e^x \)

\[ f'(x) = \frac{d}{dx}(-1) + \frac{d}{dx}((x - 1)^2 e^x) \]

\[ = 0 + (x - 1)^2 \cdot \frac{d}{dx}(e^x) + \frac{d}{dx}((x - 1)^2) e^x \quad \text{(Product Rule)} \]

\[ = (x - 1)^2 e^x + 2(x - 1) e^x \]

\[ = (x - 1)e^x(x - 1 + 2) = (x - 1)(x + 1)e^x. \]

Setting \( f'(x) = 0 \), we get \( x = 1, -1 \). [Note that \( e^x \) is never zero.]

\( f(1) = -1 \) and \( f(-1) = -1 + (-1 - 1)^2 e^{-1} = -1 + \frac{4}{e}. \)

From the figure, it is now clear that the minimum is \((1, -1)\) and the maximum is \((-1, -1 + \frac{4}{e})\).

• 4.4 #46 We have \( v = K \ln(x/x_0) = 300 \ln(x/7) \)

Solution (a) Since \( \ln x = 0 \) implies \( x = 1 \), the wind velocity is zero when \( x = 7 \), that is, at 7 centimeters.

Solution (b) Solve \( 1200 = 300 \ln(x/7) \). Dividing by 300, we get \( 4 = \ln(x/7) \). So \( x/7 = e^4 \). So \( x = 7e^4 = 38.2. \)

• Web Find \( \frac{dy}{dx} \) when \( xe^{-2y} + y \cdot e^{-3x} = \sqrt{xy} \)

Solution We take derivatives of both sides with respect to \( x \),

\[ \frac{d}{dx}(x) \cdot e^{-2y} + x \cdot \frac{d}{dx}(e^{-2y}) + \frac{d}{dx}(y) \cdot e^{-3x} + y \cdot \frac{d}{dx}(e^{-3x}) = \frac{d}{dx}[(xy)^{1/2}] \]

We used the product rule twice on the left handside, which can be simplified to

\[ e^{-2y} - 2xe^{-2y} \frac{dy}{dx} + \frac{dy}{dx} \cdot e^{-3x} - 3ye^{-3x} \]

For the righthandside, we use chain rule and product rule.

\[ \frac{d}{dx}[(xy)^{1/2}] = 1/2(xy)^{-1/2} \frac{d}{dx}[xy] = 1/2(xy)^{-1/2}[y + x \frac{dy}{dx}] \]
Solving for $\frac{dy}{dx}$, we get

$$[e^{-3x} - 2xe^{-2y} - \frac{x}{2} (xy)^{-1/2}] \frac{dy}{dx} = 3ye^{-3x} - e^{-2y} + \frac{1}{2} (xy)^{-1/2}$$

Therefore,

$$\frac{dy}{dx} = \frac{3ye^{-3x} - e^{-2y} + \frac{1}{2} (xy)^{-1/2}}{e^{-3x} - 2xe^{-2y} - \frac{1}{2} (xy)^{-1/2}}$$