The Cost of Accurate Numerical Linear Algebra

or

Can we evaluate polynomials accurately?

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Outline

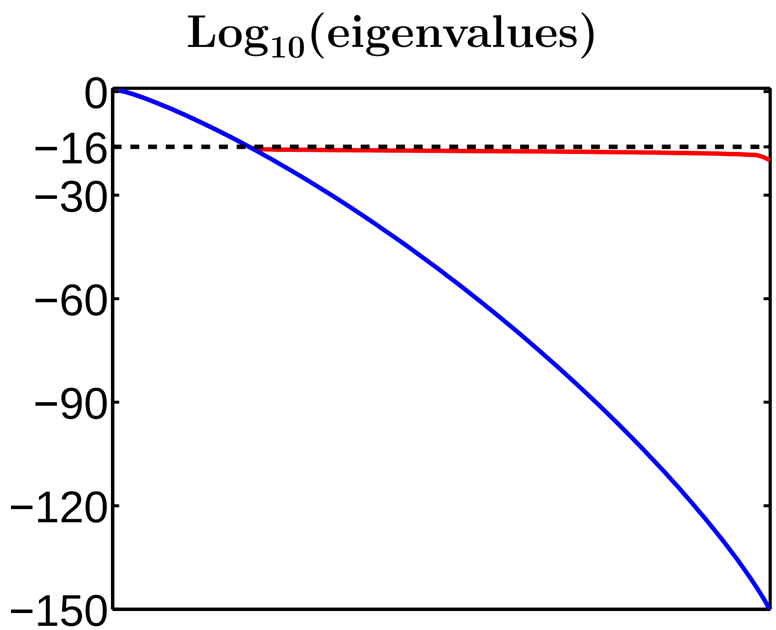
1. Motivation and Goals
2. What we can do in Traditional Model (TM) of arithmetic
3. What these example have in common:
   a condition for accurate evaluation in TM

Goal

- Compute $y = f(x)$ with floating point data x accurately and efficiently
- $f(x)$ may be
  - Rational function
  - Solution of linear system $Ay = b$
  - Solution of eigenvalue problem $Ay = \lambda y$ ...
- Accurately means with guaranteed relative error $e < 1$
  - $|y_{\text{computed}} - y| \le e \cdot |y|$
  - $e = 10^{-2}$ means 2 leading digits of y_{computed} correct
  - $y_{\text{computed}} = 0 = y$ must be exact
- Efficiently means in "polynomial time"
- Abbreviation: CAE means "Compute Accurately and Efficiently"

Example: 100 by 100 Hilbert Matrix $H(i,j) = 1/(i+j-1)$

- Eigenvalues range from 1 down to 10^{-150}
- Old algorithm, New Algorithm, both in 16 digit arithmetic



- Cost of Old algorithm in high enough precision $= O(n^3 D^2)$ where $D = \#\text{ digits} = \log(\lambda_{\max}/\lambda_{\min}) = \log \text{cond}(A) = 150$ decimal digits
- Cost of New algorithm $= O(n^3 \log D)$
- When D large, new algorithm exponentially faster
- New algorithm exploits structure of Cauchy matrices

Example: Adding Numbers in Traditional Model of Arithmetic

- $fl(a \otimes b) = (a \otimes b)(1 + \delta)$ where roundoff error $|\delta| \leq \epsilon \ll 1$
- How can we lose accuracy?
  - OK to multiply, divide, add positive numbers
  - OK to subtract exact numbers (initial data)
  - Accuracy may only be lost when subtracting approximate results:

```
   .12345xxx
 - .12345yyy
 -----------
   .00000zzz
```

- Thm: In Traditional Model it is impossible to add $x + y + z$ accurately
  - Proof sketch later
- Adding numbers represented as bits easier ...
  - Later

Structure of Results (1)

- Classes of rational expressions (matrices whose entries are expressions) that we can CAE depends strongly on Model of FP Arithmetic

  1. Traditional Model (TM for short):
     $fl(a \otimes b) = (a \otimes b)(1 + \delta)$ where $|\delta| \leq \epsilon \ll 1$
     no over/underflow
  2. Bit model: inputs are $m \cdot 2^e$, with "long exponents" e (LEM for short)
  3. Bit model: inputs are $m \cdot 2^e$, with "short exponents" e (SEM for short)

  4. Other models have been proposed (not today)
     (a) Blum/Shub/Smale
     (b) Cucker/Smale
     (c) Pour-El/Richards

Structure of Results (2)

- Classes of expressions (matrices) that we can CAE are described by factorizability properties of expressions (minors of matrices)

$$\mathbf{TM} \underset{\neq}{\subset} \mathbf{LEM} \underset{\neq ?}{\subset} \mathbf{SEM}$$

- New algorithms can be exponentially faster than conventional algorithms that just use high enough precision
- Cost(CAE in LEM) related to Cost(using symbolic computing)
- Cost(CAE in SEM) related to Cost(using integers)

Structure of Results (2)

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- New algorithms can be exponentially faster than conventional algorithms that just use high enough precision
- Cost(CAE in LEM) related to Cost(using symbolic computing)
- Cost(CAE in SEM) related to Cost(using integers)

- New results:
  - Necessary condition on polynomials for existence of algorithm for accurate evaluation in TM model
  - (Conjecture from ICM 2002 wrong)

Central Role of Minors

- Being able to CAE det(A) is necessary for CAE
  - $A = LDU$ with pivoting
  - $A = QR$
  - Eigenvalues λ_i of A ...
    - * Proof: $\det(A) = \pm \prod_i D_{ii} = \pm \prod_i R_{ii} = \prod_i \lambda_i = \cdots$
- Being able to CAE all minors of A is sufficient for CAE
  - A^{-1}
    - * Proof: Cramer's rule, only need $n^2 + 1$ minors
  - $A = LDU$ with pivoting
    - * Proof: Each entry of L, D, U a quotient of minors; $O(n^3)$ needed
  - Singular values of A (Square roots of eigenvalues of $A^T A$)
    - * Proof: $A = LDU$ with complete pivoting, then SVD of LDU
  - Eigenvalues of Totally Positive matrices (Koev)
- Similar result for pseudoinverse via minors of $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}$, etc.
- Examine which expressions (minors) we can CAE

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Cost of Accuracy in TM (1)

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP	EVD
Cauchy									
TP Cauchy									
Vandermonde									
TP Vandermonde									
Confluent Vandermonde									
TP Confluent Vandermonde									
Vandermonde 3 Term Orth. Poly.									
Generalized Vandermonde									
TP Generalized Vandermonde									
Any TP									

GENP/PP/CP = Gaussian Elimination with No/Partial/Complete Pivoting
SVD = Singular Value Decomposition
NENP = Neville Elimination (bidiagonal factorization) with No Pivoting
EVD = Eigenvalue Decomposition

Cost of Accuracy in TM (2)

TP = Totally Positive (all minors nonnegative)

Matrix Type	
Cauchy	$C_{ij} = 1/(x_i + y_j)$
TP Cauchy	$x_i \nearrow$, $y_j \nearrow$, $x_1 + y_1 > 0$
Vandermonde	$V_{ij} = x_i^{j-1}$, x_i distinct
TP Vandermonde	$0 < x_i \nearrow$
Confluent Vandermonde	if some x_i coincide, differentiate rows of V
TP Confluent Vandermonde	$0 < x_i \nearrow$
Vandermonde 3 Term Orth. Poly.	$V_{ij} = P_j(x_i)$, P_j orthogonal polynomial from 3-term recurrence
Generalized Vandermonde	$G_{ij} = x_i^{\lambda_j + j - 1}$, λ_j nonnegative increasing integer sequence
TP Generalized Vandermonde	$0 < x_i \nearrow$
Any TP	Given by its Neville Factorization

Cost of Accuracy in TM
Known results + New Results

Matrix Type	det(A)	A^{-1}	Minor	GENP	GEPP	GECP	SVD	NENP	EVD
Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2	
TP Cauchy	n^2	n^2	n^2	n^2	n^2	n^3	n^3	n^2	n^3
Vandermonde	n^2	No	No	No	No	No	n^3	n^2	
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^3
Confluent Vandermonde	n^2	No	No	No	No	No		n^2	
TP Confluent Vandermonde	n^2	n^3		n^3			n^3	n^2	n^3
Vandermonde 3 Term Orth. Poly.	n^2						n^3		
Generalized Vandermonde	No	No	No	No	No	No		No	
TP Generalized Vandermonde	Λn^2	Λn^3	exp	Λn^2	Λn^2	exp	Λn^3	Λn^2	Λn^3
Any TP	n	n^3	exp	n^3	exp	exp	n^3	0	n^3

Other examples in Traditional Model

- Diagonal * Totally Unimodular (TU) * Diagonal
  - TU $\Leftrightarrow$ each minor $\in \{0, \pm 1\}$
  - Poincaré: Signed incidence matrix on graph $\Rightarrow$ TU
  - Includes 2nd centered difference approximations to Sturm-Liouville equations and elliptic PDEs on uniform meshes
  - One-line change to GECP makes it accurate, then SVD, EVD
- Sparse matrices with
  - Acyclic sparsity patterns, GECP cost $= O(n^3)$
  - Particular sparsity and sign patterns ("Total Sign Compound") GECP Cost $= O(n^4)$
- Weakly Diagonally Dominant (WDD) M-Matrices
  - M-matrix: off-diagonal $A_{ij} < 0$, all $(A^{-1})_{ij} > 0$
  - WDD: nonnegative row sums $s_i = \sum_j A_{ij} \geq 0$
  - Modify GECP to update s_i, off-diagonal A_{ij}, cost $= O(n^3)$
- What do these examples have in common?

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What do all these examples have in common?

- Recall models of computation:
  - Traditional Model (TM):
    $fl(a \otimes b) = (a \otimes b)(1+\delta)$ where $|\delta| \le \epsilon < 1$, δ real
  - Long Exponent Model (LEM): inputs are $m \cdot 2^e$, with "long" e
  - Short Exponent Model (SEM): inputs are $m \cdot 2^e$, with "short" e
- Goals: Given choice of model
  - Decide if $\exists$ algorithm $alg(x, \delta)$ to evaluate multivariate polynomial $p(x)$ with small relative error on domain $\mathcal{D}$:

    $\forall\ 0 < \eta < 1$ ... η = desired relative error

    $\exists\ 0 < \epsilon < 1$ ... ϵ = maximum rounding error

    $\forall\ x \in \mathcal{D}$ ... for all x in the domain

    $\forall\ |\delta_i| \le \epsilon$ ... for all rounding errors bounded by ϵ

    $|alg(x,\delta) - p(x)| \le \eta \cdot |p(x)|$ ... relative error is at most η
  - If so, is there a polynomial-time algorithm?
  - Given $p(x)$ and $\mathcal{D}$, seek effective procedure to exhibit algorithm, or show one does not exist

What is known about existence of accurate algorithms?

- Depends on
  - Choice of model (TM, LEM, SEM)
  - TM needs more details to be formal
  - How $p(x)$ presented (explicit, determinant, ...)
- Existence of accurate algorithm
  - Bit Models (LEM and SEM): An accurate algorithm always exists
  - TM: may or may not exist
    We show current progress towards a decision procedure
- Existence of polynomial-time accurate algorithm
  - TM $\subsetneq$ LEM $\underset{\neq ?}{\subseteq}$ SEM

Formalizing an Algorithm under Traditional Model

- **Numerical operations included**
  - Could include $\pm$, $\times$, $\div$, unary $-$, ...
  - We omit $\div$ (restrictive?)
  - We say unary $-$ is exact (true in practice)
- **Comparison and Branching**
  - Assume branching on exact comparisons $a > b$, $c \leq d$, ...
  - Will sketch proof in nonbranching case
- **Determinism**
  - Is $3 + 7$ same no matter where computed?
  - Will assume nondeterministic for now (try to include later...)
- **Available constants**
  - With $\sqrt{2}$, could compute $x^2 - 2 = (x - \sqrt{2}) \times (x + \sqrt{2})$ accurately, else not
  - Will sketch proof when no constants
  - Limits us to integer coefficients, zero constant term in $p(x)$
    - Replace $2 \times x$ by $x + x$, etc.
    - No loss of generality for homogeneous polynomials, integer coeffs

Recognizing Accuracy

- Ex: Compute $p(x) = x_1 + x_2 + x_3$
  - Try $alg(x, \delta) = ((x_1 + x_2)(1 + \delta_1) + x_3)(1 + \delta_2)$
  - $rel\_err(x, \delta) = \frac{alg(x,\delta) - p(x)}{p(x)} = \frac{x_1 + x_2}{x_1 + x_2 + x_3}(\delta_1 + \delta_2 + \delta_1 \cdot \delta_2) + \frac{x_3}{x_1 + x_2 + x_3}(\delta_2)$
  - $\forall \epsilon > 0$, $rel\_err(x, \delta)$ unbounded on an open subset of (x, δ) with $|\delta_i| < \epsilon$
- Generally: $rel\_err(x, \delta) = \sum_r \frac{p_r(x)}{p(x)} \cdot q_r(\delta)$
  - Each $\frac{p_r(x)}{p(x)}$ must be bounded near $p(x) = 0$
- Ex: $p(x)$ positive definite and homogeneous, degree d
  - If $p_r(x)$ also homogeneous, degree d, then $\frac{p_r(x)}{p(x)}$ bounded
  - Holds if all intermediate results are homogeneous

Examples

- $M_2(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 2 \cdot z^2)$
  - Positive definite and homogenous, easy to evaluate accurately
- $M_3(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 3 \cdot z^2)$
  - Motzkin polynomial, nonnegative, zero at $|x| = |y| = |z|$

$$
\begin{array}{ll}
\text{if} & |x-z| \le |x+z| \wedge |y-z| \le |y+z| \\
p = & z^4 \cdot [4((x-z)^2 + (y-z)^2 + (x-z)(y-z))] + \\
& +z^3 \cdot [2(2(x-z)^3 + 5(y-z)(x-z)^2 + 5(y-z)^2(x-z) + \\
& \qquad 2(y-z)^3)] + \\
& +z^2 \cdot [(x-z)^4 + 8(y-z)(x-z)^3 + 9(y-z)^2(x-z)^2 + \\
& \qquad 8(y-z)^3(x-z) + (y-z)^4] + \\
& +z \cdot [2(y-z)(x-z)((x-z)^3 + 2(y-z)(x-z)^2 + \\
& \qquad 2(y-z)^2(x-z) + (y-z)^3] + \\
& +(y-z)^2(x-z)^2((x-z)^2 + (y-z)^2) \\
\text{else} & \ldots\ 2^{\#\text{vars}-1} \text{ more analogous cases}
\end{array}
$$

- $M_4(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 4 \cdot z^2)$
  - Impossible to evaluate accurately

Allowable Sets

- Define basic *allowable sets*
  - $Z_i = \{x : \ x_i = 0\}$
  - $S_{ij} = \{x : \ x_i + x_j = 0\}$
  - $D_{ij} = \{x : \ x_i - x_j = 0\}$
- Def: A set is *allowable* if it can be written as an arbitrary union and intersection of basic allowable sets (plus null set, R^n)
- Def: Allow(x) is the smallest allowable set containing x

$$\mathrm{Allow}(x) = \mathrm{R}^n \cap (\cap_{i:\ x_i=0} Z_i) \cap (\cap_{i,j:\ x_i+x_j=0} S_{ij}) \cap (\cap_{i,j:\ x_i-x_j=0} D_{ij})$$

- Ex: $\mathrm{Allow}((0, 1, -1, 2)) = Z_1 \cap S_{23}$
- We say $p(x)$ allowable if its variety $V(p)$ is allowable
- If $p(x)$ not allowable, then

$$G(p) \equiv V(p) - \cup A$$

  is nonempty, where the union is over all allowable sets A contained in $V(p)$
- Def: $G(p)$ called the set of points in "general position" in $V(p)$

Existence of an Accurate Algorithm in TM

- Consider algorithms that
  - Include $\pm$, $\times$, branching
  - $\pm$ and $\times$ incur $1+\delta$ errors
  - Comparisons and unary negation are exact
  - No branching on η, ϵ
  - No explicit constants (limits results to integer coefficients, no constant term)
  - Nondeterministic rounding errors
  - Domain $\mathcal{D} = \mathrm{R}^n$
- Theorem: A necessary condition for the existence of an accurate algorithm to evaluate $p(x)$ on R^n is that $V(p)$ be allowable.

Examples

- $p(x, y, z) = x + y + z$ not allowable (D., Koev)
- $M_2(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 2 \cdot z^2)$ is allowable: $V(M_2) = \{0\}$
- $M_3(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 3 \cdot z^2)$ is allowable: $V(M_3) = \{|x| = |y| = |z|\}$.
- $M_4(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 4 \cdot z^2)$ is unallowable
- Allowable $V(p)$ *not* a sufficient condition for an accurate algorithm: $p(x, y, z, w) = w^4 + w^2 \cdot (x + y + z)^2$ has allowable $V(p) = \{w = 0\}$, but (apparently) can't be evaluated accurately

Proof Sketch (1):

- Assume no branching for simplicity
- Let $alg(x, \delta)$ denote result of any computation.
- Main Lemma: Choose any x. One of following two cases must hold:
  1. $alg(x, \delta)$ is nonzero at x for all δ in a Zariski-open set
  2. $alg(y, \delta) = 0$ for all $y \in \text{Allow}(x)$ and all δ
- Suppose $V(p)$ not allowable. Choose any $x \in G(p) \subset V(p)$. Then either
  1. $alg(x, \delta)$ is nonzero at x for all δ in a Zariski-open set
     but $p(x) = 0$, so the relative error is ∞
  2. $alg(y, \delta) = 0$ for all $y \in \text{Allow}(x)$ and all δ
     but $p(y) \neq 0$ a.e., so the relative error is 1

Proof Sketch (2): Main Lemma

- Main Lemma: Choose any x. One of following two cases must hold:
  1. $alg(x, \delta)$ is nonzero at x for all δ in a Zariski-open set
  2. $alg(y, \delta) = 0$ for all $y \in \text{Allow}(x)$ and all δ
- For simplicity, suppose no branching, no data reuse, nondeterminism
  - Implies that $alg(x, \delta)$ can be represented as a graph:
    * Source nodes representing data x_i, output edges connected to ...
    * Computational nodes, arranged in a tree, of following kinds:
      · 2-inputs, producing $fl(a \otimes b) = (a \otimes b)(1 + \delta_{\text{node}})$ $(\otimes \in \{+, -, \times\})$ with independent $|\delta_{\text{node}}| \leq \epsilon$ for each node
      · 1-input, producing $fl(x \otimes x) = (x \otimes x)(1 + \delta_{\text{node}})$ (note: $fl(x - x) = 0$ exactly)
      · 1-input, producing $-x$ exactly
    * Destination node, one input, no output

Proof Sketch (3): Main Lemma (cont)

- Main Lemma: Choose any x. One of following two cases must hold:

  1. $alg(x, \delta)$ is nonzero at x for all δ in a Zariski-open set
  2. $alg(y, \delta) = 0$ for all $y \in \text{Allow}(x)$ and all δ

- Def: Choose x. Call computational node "nontrivial" if it

  – Computes $fl(a \pm b)$, both a and b nonzero as polynomials in δ

  – At least one of a and b not an input x_i

- Lemma: Output of all nontrivial nodes nonzero on Zariski-open set of δ
- If ultimate output is from nontrivial node, done (Case 1)
- Otherwise, "trace back" zero output through tree as far as possible
- Can show (case analysis) that zero must result from one of

  – $x_i = 0$ (allowable)

  – $x_i \pm x_j = 0$ (allowable)

  – $x - x$ or $x + (-x)$ (in which case $alg(x, \delta) \equiv 0$)

- In any case, $alg(y, \delta)$ must be zero on Allow(x) (Case 2)

Other results and Future Work

- Large relative error, if it occurs, occurs on open set of (x, δ)
  - So hard problems not of measure zero
- Want to incorporate
  - Determinism (simulate deterministic machine by nondeterministic one)
  - Constants (add $\{x : x_i \pm \alpha = 0\}$ to basic allowable sets for constant α)
  - Domain $\mathcal{D}$ limited to (allowable?) semialgebraic sets
  - Division and rational functions
- Complete decision procedure, just not necessary or sufficient conditions
  - Since $p(x) = x_1^{2n} + x_1^2 \cdot (q(x_2, .., x_n))^2$ has $V(p) = \{x : x_1 = 0\}$, behavior of $q()$ "hidden"
  - Need to inductively "unfold" $V(p)$
- Extend to complex arithmetic, interval arithmetic
- Perturbation theory
  - Conj: Accurate evaluation possible iff condition number can have certain singularities

In Contrast: Adding Numbers in Bit Model of Arithmetic

- $x = m \cdot 2^e$ where m and e are integers, m at most b bits
- $fl(x+y)$ is correctly rounded result
- Cancellation is obstable to accuracy:
  - $(2^e + 1) - 2^e$ requires e bits of intermediate precision
  - Not polynomial time!
- "Sort and Sum" Algorithm for $S = \sum_{i=1}^{n} x_i$

  Sort so $|e_1| \geq |e_2| \geq \cdots \geq |e_n|$ ... $|x_1| \geq \cdots \geq |x_n|$ more than enough
  $S = 0$ ... $B > b$ bits
  for $i = 1$ to n
  $\quad S = S + x_i$

- Thm: Let $N = 1 + 2^{B-b} + 2^{B-2b} + \cdots 2^{B \bmod b} = 1 + \lceil \frac{2^{B-b}}{1-2^{-b}} \rceil$. Then
  - If $n \leq N$, then S accurate to nearly b bits, despite any cancellation
  - If $n \geq N + 2$, then S may be completely wrong (wrong sign)
  - If $n = N + 1$, more cases ...
- Ex: x_i double ($b = 53$), S extended ($B = 64$) $\Rightarrow N = 2049$

Conclusions

- **We have identified many classes of floating point expressions and matrix computations that permit**
  - **Accurate solutions: relative error < 1**
  - **Efficient solutions: time = poly(input size)**
- **Explored 3 natural models of arithmetic**
  - **Traditional Model (TM)**
  - **Long Exponent Model (LEM)**
  - **Short Exponent Model (SEM)**
- **New efficient algorithms for each: TM $\subsetneq$ LEM $\subsetneq ?$ SEM**
- **New necessary condition for existence of accurate algorithm to evaluate $p(x)$ in TM – working on effective decision procedure**
- **Lots of open problems**
- **For more information see**
  - `www.cs.berkeley.edu/~demmel`
  - `math.mit.edu/~plamen`

Extra Slides

Traditional Model - What we can do

- **What do all these examples have in common?**
- **Goal: evaluate homogeneous polynomial $f(x)$ accurately on domain $\mathcal{D}$**
- **Property A: $f = \Pi_m f_m$ where each factor f_m satisfies one of**
  1. **f_m of the form x_i, $x_i - x_j$ or $x_i + x_j$, or**
  2. **$|f_m|$ bounded away from 0 on $\mathcal{D}$**
- **Conjecture 1: f satisfies Prop. A iff $f(x)$ can be evaluated accurately**
- **Conjecture 2: f satisfies Prop. A iff $f(x)$ has a *relative perturbation theory*:**
  - **relative error in output = O($\kappa_{rel}\cdot$ relative error in input)**
  - **$\kappa_{rel} = O(1/\min \frac{|x_i \pm x_j|}{|x_i|+|x_j|}) = O(1/$ smallest relative gap among inputs)**
  - **Tiny outputs often well conditioned**
  - **Relative perturbation theory justifies computing them!**

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  - **Tiny outputs often well conditioned**
  - **Relative perturbation theory justifies computing them!**
- **WRONG**
  - **Conjecture only true in "if" direction**
  - **$w^4 + w^2 \cdot (x+y)^2$ ok**
  - **$w^4 + w^2 \cdot (x+y+z)^2$ not ok**
  - **Both irreducible with same real variety $\{w = 0\}$**

Bit Models of Arithmetic

- Inputs of form $x = m \cdot 2^e$, e and m integers
- size(x) = # bits used to represent x = #bits(m) + #bits(e)
- Can evaluate any rational expression accurately
  - Convert to poly/poly, using high enough precision
  - Question is cost
- Cost depends strongly on #bits(e)
  - Short Exponent Model (SEM)
    * #bits(e) = O(log(#bits(m)))
    * Equivalent to integer arithmetic
    * Can CAE many problems
  - Long Exponent Model (LEM)
    * #bits(e) and #bits(m) independent
    * Natural model for algorithm design
    * Like symbolic algebra, which is much harder

Differences between Short and Long Exponent Models

- **SEM and integer arithmetic "equivalent"**
  - **Represent $m \cdot 2^e$ as integer with**
    $\#\text{bits} = \#\text{bits}(m) + e \approx \#\text{bits}(m) + 2^{\#\text{bits}(e)} = \text{poly}(\#\text{bits}(m))$
  - **Any minor of any SEM matrix A computable accurately in poly time**
    - **Use Clarkson's Algorithm**
  - **Can do accurate linear algebra in polynomial time**
- **LEM and integer arithmetic not equivalent**
  - **$\Pi_{i=1}^{n}(1 + x_i)$ can have exponentially more bits if x_i LEM than SEM**
  - **Getting arbitrary bit of $\Pi_{i=1}^{n}(1 + x_i)$ as hard as permanent**
  - **Testing if an LEM matrix is singular may not be in NP**
  - **For efficiency, matrices need structure**
- **Cond(A) in LEM can be exponentially larger than in SEM**
  - **SEM: $\log \text{cond}(A)$ is poly(size(A))**
  - **LEM: $\log \text{cond}(A)$ can be exponential in size(A)**

Which FP Expressions can we CAE in the Long Exponent Model (LEM)?

- Def: $r(x)$ is in factored form if it is written as explicit product of sparse polynomials
  - E.g.: *not* as determinant of general matrix
- Def: size(r) = #bits to write down r
- Theorem: We can CAE r in time poly(size(r))
  - Compute monomials in each sparse polynomial exactly
  - Add them in decreasing order by magnitude, with rounding (see Hida's talk)
- Def: A family $A_n(x)$ of n-by-n rational matrices is polyfactorable if each minor $r(x)$ is in factored form of size size(r) = O(poly(n))
- Theorem: Suppose $A_n(x)$ is polyfactorable. Then in the LEM we can CAE LU with pivoting, A^{-1}, singular values.

Summary of differences between Arithmetic Models

- **What can we CAE in LEM that we could not in TM?**
  - **Rational Expressions**
    - **LEM: anything in factored form**
    - **TM: not $x + y + z$ or any expression with nontrivial real variety**
  - **Matrix computations: polyfactorable matrices**
    - **Take any $A(x)$ that we can CAE in TM, substitute $x_i = \text{poly}_i(y_1, ..., y_n)$**
    - **Green's matrices (inverses of tridiagonals, represented as $A_{ij} = x_i y_j$)**
- **What can we CAE in SEM that we could not in LEM?**
  - **$\det(A)$ where each A_{ij} is a general floating point number**

Open Questions

- Are there FP expressions that we provably cannot CAE in LEM?
  - $\Pi_{i=1}^{n}(1 + x_i) - \Pi_{j=1}^{n}(1 + y_j)$
  - Determinant of general (or just tridiagonal) matrix
- What changes if we have sign information?
  - We have accurate algorithms for all TP matrices, but not efficient
  - How big a class of TP matrices can we do efficiently?
    (see Koev's talk)
- Differential equations
  - Only simplest ones understood (eg M-matrices)
  - What about other discretizations?
  - Conjecture: Accuracy depends only on geometry, not material properties
- Accuracy of singular vectors, eigenvectors
- What about nonsymmetric eigenproblem?

Conclusions

- **We have identified many classes of floating point expressions and matrix computations that permit**
  - **Accurate solutions: relative error < 1**
  - **Efficient solutions: time = poly(input size)**
- **Explored 3 natural models of arithmetic**
  - **Traditional Model (TM)**
  - **Long Exponent Model (LEM)**
  - **Short Exponent Model (SEM)**
- **New efficient algorithms for each**
- **TM $\subsetneq$ LEM $\overset{\subseteq}{\neq ?}$ SEM**
- **Lots of open problems**
- **For more information see**
  - `www.cs.berkeley.edu/~demmel`
  - `www-math.mit.edu/~koev`