# The Cost of Accurate Numerical Linear Algebra

# or

# Can we evaluate polynomials accurately?

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**James Demmel**
**Mathematics and Computer Science**
**UC Berkeley**

**Joint work with**
**Ioana Dumitriu, Olga Holtz**

**Plamen Koev, Yozo Hida, Ben Diament**
**W. Kahan, Ming Gu, Stan Eisenstat, Ivan Slapničar,**
**Krešimir Veselić, Zlatko Drmač**



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# Outline

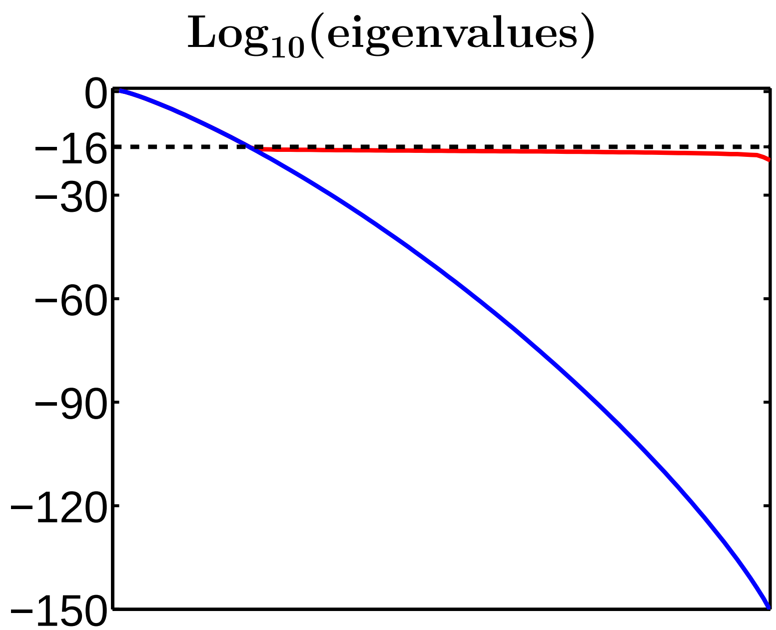
1. Motivation and Goals
2. What we can do in Traditional Model (TM) of arithmetic
3. What these example have in common:
   a condition for accurate evaluation in TM

# Goal

- Compute $y = f(x)$ with floating point data $x$ accurately and efficiently
- $f(x)$ may be
  - Rational function
  - Solution of linear system $Ay = b$
  - Solution of eigenvalue problem $Ay = \lambda y$ ...
- Accurately means with guaranteed relative error $e < 1$
  - $|y_{\text{computed}} - y| \le e \cdot |y|$
  - $e = 10^{-2}$ means 2 leading digits of $y_{\text{computed}}$ correct
  - $y_{\text{computed}} = 0 = y$ must be exact
- Efficiently means in "polynomial time"
- Abbreviation: CAE means "Compute Accurately and Efficiently"

# Example: 100 by 100 Hilbert Matrix $H(i,j) = 1/(i+j-1)$

- Eigenvalues range from 1 down to $10^{-150}$
- Old algorithm, New Algorithm, both in 16 digit arithmetic




- Cost of Old algorithm in high enough precision $= O(n^3 D^2)$ where $D = \#$ digits $= \log(\lambda_{\max}/\lambda_{\min}) = \log \mathrm{cond}(A) = 150$ decimal digits
- Cost of New algorithm $= O(n^3 \log D)$
- When $D$ large, new algorithm exponentially faster
- New algorithm exploits structure of Cauchy matrices

## Example: Adding Numbers in Traditional Model of Arithmetic

- $fl(a \otimes b) = (a \otimes b)(1 + \delta)$ where roundoff error $|\delta| \leq \epsilon \ll 1$
- How can we lose accuracy?
  - OK to multiply, divide, add positive numbers
  - OK to subtract exact numbers (initial data)
  - Accuracy may only be lost when subtracting approximate results:

$$\begin{array}{rl} & .12345\text{xxx} \\ - & .12345\text{yyy} \\ \hline & .00000\text{zzz} \end{array}$$

- Thm: In Traditional Model it is impossible to add $x + y + z$ accurately
  - Proof sketch later
- Adding numbers represented as bits easier ...

# Adding Numbers in Bit Model of Arithmetic

- $x = m \cdot 2^e$ where $m$ and $e$ are integers, $m$ at most $b$ bits
- $fl(x+y)$ is correctly rounded result
- Cancellation is obstable to accuracy and efficiency:
  - $(2^e + 1) - 2^e$ requires $e$ bits of intermediate precision
  - Not polynomial time!
- "Sort and Sum" Algorithm for $S = \sum_{i=1}^{n} x_i$

  Sort so $|e_1| \geq |e_2| \geq \cdots \geq |e_n|$ ... $|x_1| \geq \cdots \geq |x_n|$ more than enough
  $S = 0$ ... $B > b$ bits
  for $i = 1$ to $n$
  $\quad S = S + x_i$

- Thm: Let $N = 1 + 2^{B-b} + 2^{B-2b} + \cdots 2^{B \bmod b} = 1 + \lceil \frac{2^{B-b}}{1-2^{-b}} \rceil$. Then
  - If $n \leq N$, then $S$ accurate to nearly $b$ bits, despite any cancellation
  - If $n \geq N+2$, then $S$ may be completely wrong (wrong sign)
  - If $n = N+1$, 2 cases, depending on whether $s_2$ denormal
- Ex: $x_i$ double ($b = 53$), $S$ extended ($B = 64$) $\Rightarrow N = 2049$

## Structure of Prior Results

- Classes of rational expressions (matrices whose entries are expressions) that we can CAE depends strongly on Model of FP Arithmetic
  1. Traditional Model (TM for short):
     $fl(a \otimes b) = (a \otimes b)(1 + \delta)$ where $|\delta| \leq \epsilon \ll 1$
     no over/underflow
  2. Bit model: inputs are $m \cdot 2^e$, with "long exponents" $e$ (LEM for short)
  3. Bit model: inputs are $m \cdot 2^e$, with "short exponents" $e$ (SEM for short)
- Classes of expressions (matrices) that we can CAE are described by factorizability properties of expressions (minors of matrices)

$$\text{TM} \subsetneq \text{LEM} \underset{\neq ?}{\subseteq} \text{SEM}$$

- New algorithms can be exponentially faster than conventional algorithms that just use high enough precision

# Structure of New Results

- All in Traditional Model (TM): $fl(a \otimes b) = (a \otimes b)(1 + \delta)$ where $|\delta| \leq \epsilon \ll 1$
- Necessary condition on polynomial $p(x)$ for existence of algorithm for accurate evaluation in TM model
  - Just depends on variety $V(p) = \{x : \; p(x) = 0\}$
  - Conjecture from ICM 2002 half right - not sufficient in real case!
- Goal: decision procedure to either exhibit accurate algorithm for $p$, or proof that one does not exist
  - When data complex: Simple necc. & suff. condition on $V(p)$
  - When data real: Will show main induction step

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## Cost of Accuracy in TM (1)

| Matrix Type | det($A$) | $A^{-1}$ | Minor | GENP | GEPP | GECP | SVD | NENP | EVD |
|---|---|---|---|---|---|---|---|---|---|
| Cauchy | | | | | | | | | |
| TP Cauchy | | | | | | | | | |
| Vandermonde | | | | | | | | | |
| TP Vandermonde | | | | | | | | | |
| Confluent Vandermonde | | | | | | | | | |
| TP Confluent Vandermonde | | | | | | | | | |
| Vandermonde 3 Term Orth. Poly. | | | | | | | | | |
| Generalized Vandermonde | | | | | | | | | |
| TP Generalized Vandermonde | | | | | | | | | |
| Any TP | | | | | | | | | |

**GENP/PP/CP = Gaussian Elimination with No/Partial/Complete Pivoting**
**SVD = Singular Value Decomposition**
**NENP = Neville Elimination (bidiagonal factorization) with No Pivoting**
**EVD = Eigenvalue Decomposition**

## Cost of Accuracy in TM (2)

**TP = Totally Positive (all minors nonnegative)**

| Matrix Type | |
|---|---|
| Cauchy | $C_{ij} = 1/(x_i + y_j)$ |
| TP Cauchy | $x_i \nearrow$, $y_j \nearrow$, $x_1 + y_1 > 0$ |
| Vandermonde | $V_{ij} = x_i^{j-1}$, $x_i$ distinct |
| TP Vandermonde | $0 < x_i \nearrow$ |
| Confluent Vandermonde | if some $x_i$ coincide, differentiate rows of $V$ |
| TP Confluent Vandermonde | $0 < x_i \nearrow$ |
| Vandermonde 3 Term Orth. Poly. | $V_{ij} = P_j(x_i)$, $P_j$ orthogonal polynomial from 3-term recurrence |
| Generalized Vandermonde | $G_{ij} = x_i^{\lambda_j+j-1}$, $\lambda_j$ nonnegative increasing integer sequence |
| TP Generalized Vandermonde | $0 < x_i \nearrow$ |
| Any TP | Given by its Neville Factorization |

## Cost of Accuracy in TM

## Known results + New Results

| Matrix Type | det($A$) | $A^{-1}$ | Minor | GENP | GEPP | GECP | SVD | NENP | EVD |
|---|---|---|---|---|---|---|---|---|---|
| Cauchy | $n^2$ | $n^2$ | $n^2$ | $n^2$ | $n^2$ | $n^3$ | $n^3$ | $n^2$ | |
| TP Cauchy | $n^2$ | $n^2$ | $n^2$ | $n^2$ | $n^2$ | $n^3$ | $n^3$ | $n^2$ | $n^3$ |
| Vandermonde | $n^2$ | No | No | No | No | No | $n^3$ | $n^2$ | |
| TP Vandermonde | $n^2$ | $n^3$ | exp | $n^2$ | $n^2$ | exp | $n^3$ | $n^2$ | $n^3$ |
| Confluent Vandermonde | $n^2$ | No | No | No | No | No | | $n^2$ | |
| TP Confluent Vandermonde | $n^2$ | $n^3$ | | $n^3$ | | | $n^3$ | $n^2$ | $n^3$ |
| Vandermonde 3 Term Orth. Poly. | $n^2$ | | | | | | $n^3$ | | |
| Generalized Vandermonde | No | No | No | No | No | No | | No | |
| TP Generalized Vandermonde | $\Lambda \mathrm{n}^2$ | $\Lambda \mathrm{n}^3$ | exp | $\Lambda \mathrm{n}^2$ | $\Lambda \mathrm{n}^2$ | exp | $\Lambda n^3$ | $\Lambda \mathrm{n}^2$ | $\Lambda n^3$ |
| Any TP | $n$ | $n^3$ | exp | $n^3$ | exp | exp | $n^3$ | 0 | $n^3$ |

# Other examples in Traditional Model

- Diagonal * Totally Unimodular * Diagonal
  - Includes 2nd centered difference approximations to Sturm-Liouville equations and elliptic PDEs on uniform meshes
- Sparse matrices with
  - Acyclic sparsity patterns
  - Particular sparsity and sign patterns: "Total Sign Compound"
- Weakly Diagonally Dominant M-Matrices
- What do these examples have in common?

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# What do all these examples have in common?

- Notation
  - $p(x)$ is polynomial to be evaluated, $x = (x_1, x_2, ...)$
  - $p_{comp}(x, \delta)$ is result of algorithm for $p(x)$
  - $\delta = (\delta_1, \delta_2, ...)$ is vector of rounding errors
- Goal: Decide if $\exists$ algorithm $p_{comp}(x, \delta)$ to evaluate $p(x)$ with small relative error on domain $\mathcal{D}$:

  $\forall\, 0 < \eta < 1$ ... for any $\eta$ = desired relative error

  $\exists\, 0 < \epsilon < 1$ ... there is an $\epsilon$ = maximum rounding error

  $\forall\, x \in \mathcal{D}$ ... so that for all $x$ in the domain

  $\forall\, |\delta_i| \le \epsilon$ ... and for all rounding errors bounded by $\epsilon$

  $|p_{comp}(x, \delta) - p(x)| \le \eta \cdot |p(x)|$ ... relative error is at most $\eta$
- Given $p(x)$ and $\mathcal{D}$, seek effective procedure ("compiler") to exhibit algorithm, or show one does not exist
- Not obviously Tarski-decideable: how do we express "$\exists$ algorithm"?

# Formalizing an Algorithm under Traditional Model

- **Numerical operations included**
  - Include $\pm$, $\times$, (exact) unary $-$
  - We omit $\div$ (restrictive?)
- **Comparison and Branching**
  - Assume branching on exact comparisons $a > b$, $c \leq d$, ...
  - Will sketch proof in nonbranching case
- **Determinism**
  - Is $3 + 7$ same no matter where computed?
  - Will assume nondeterministic for now
- **Available constants**
  - With $\sqrt{2}$, could compute $x^2 - 2 = (x - \sqrt{2}) \times (x + \sqrt{2})$ accurately, else not
  - Will sketch proof when no constants
  - Limits us to integer coefficients, zero constant term in $p(x)$
    - Replace $2 \times x$ by $x + x$, etc.
    - No loss of generality for homogeneous polynomials, integer coeffs

# Recognizing Accuracy

- Ex: Compute $p(x) = x_1 + x_2 + x_3$
  - Try $p_{comp}(x, \delta) = ((x_1 + x_2)(1 + \delta_1) + x_3)(1 + \delta_2)$
  - $rel\_err(x, \delta) = \frac{p_{comp}(x,\delta) - p(x)}{p(x)} = \frac{x_1+x_2}{x_1+x_2+x_3}(\delta_1 + \delta_2 + \delta_1 \cdot \delta_2) + \frac{x_3}{x_1+x_2+x_3}(\delta_2)$
  - $\forall \epsilon > 0$, $rel\_err(x, \delta)$ unbounded on an open subset of $(x, \delta)$ with $|\delta_i| < \epsilon$
- Generally: $rel\_err(x, \delta) = \sum_\alpha \frac{p_\alpha(x)}{p(x)} \cdot \delta^\alpha$
  - Each $\frac{p_\alpha(x)}{p(x)}$ must be bounded near $p(x) = 0$
- Ex: $p(x) > 0$ (positive definite) and homogeneous of degree $d$
  - If $p_\alpha(x)$ also homogeneous of degree $d$, then $\frac{p_\alpha(x)}{p(x)}$ bounded
  - Holds if all intermediate results in $p_{comp}$ are homogeneous

# Examples

- $M_2(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 2 \cdot z^2)$
  - Positive definite and homogenous, easy to evaluate accurately
- $M_3(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 3 \cdot z^2)$
  - Motzkin polynomial, nonnegative, zero at $|x| = |y| = |z|$

$$
\begin{array}{rl}
\text{if} & |x - z| \leq |x + z| \wedge |y - z| \leq |y + z| \\
p = & z^4 \cdot [4((x - z)^2 + (y - z)^2 + (x - z)(y - z))] + \\
& +z^3 \cdot [2(2(x - z)^3 + 5(y - z)(x - z)^2 + 5(y - z)^2(x - z) + \\
& \qquad 2(y - z)^3)] + \\
& +z^2 \cdot [(x - z)^4 + 8(y - z)(x - z)^3 + 9(y - z)^2(x - z)^2 + \\
& \qquad 8(y - z)^3(x - z) + (y - z)^4] + \\
& +z \cdot [2(y - z)(x - z)((x - z)^3 + 2(y - z)(x - z)^2 + \\
& \qquad 2(y - z)^2(x - z) + (y - z)^3] + \\
& +(y - z)^2(x - z)^2((x - z)^2 + (y - z)^2) \\
\text{else} & \ldots\ 2^{\#\text{vars}-1} \text{ more analogous cases}
\end{array}
$$

- $M_4(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 4 \cdot z^2)$
  - Impossible to evaluate accurately

# Allowable Sets

- Define basic *allowable sets*
  - $Z_i = \{x : x_i = 0\}$
  - $S_{ij} = \{x : x_i + x_j = 0\}$
  - $D_{ij} = \{x : x_i - x_j = 0\}$
- Def: A set is *allowable* if it can be written as an arbitrary union and intersection of basic allowable sets (plus null set, $\mathrm{R}^n$)
- We say $p(x)$ is allowable if its variety $V(p)$ is allowable

## Necessary condition for existence of an Accurate Algorithm

- **Theorem: A necessary condition for the existence of an accurate algorithm to evaluate $p(x)$ on $R^n$ or $C^n$ is that $V(p)$ be allowable.**
  - **Proof sketch later (if time)**
- **Examples**
  - $p(x, y, z) = x + y + z$ **not allowable (D., Koev)**
  - $M_2(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 2 \cdot z^2)$ **is allowable:** $V(M_2) = \{0\}$
  - $M_3(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 3 \cdot z^2)$ **is allowable:** $V(M_3) = \{|x| = |y| = |z|\}$.
  - $M_4(x, y, z) = z^6 + x^2 \cdot y^2 \cdot (x^2 + y^2 - 4 \cdot z^2)$ **is unallowable**
  - $V(\det(\text{Toeplitz}))$ **is unallowable** $\Rightarrow$ **no accurate linear algebra for Toeplitz matrices in TM: need arbitrary precision arithmetic**
  - $V(\det(\text{your favorite structured matrix}))$ **...**

# Sufficient conditions for accurate evaluation

- Over $C^n$, $V(p)$ being allowable is necessary and sufficient for accuracy
  - Proof Sketch: Can show $V(p)$ allowable $\Rightarrow p = c \cdot \Pi_i p_i$ where each $p_i$ of form $x_j$ or $x_j \pm x_k$
- Over $R^n$, $V(p)$ being allowable *not* a sufficient condition for accuracy:
  - $p = (u^4 + v^4) + (u^2 + v^2)(x + y + z)^2$ and
    $q = (u^4 + v^4) + (u^2 + v^2)(x^2 + y^2 + z^2)$
    are both allowable: $V(p) = V(q) = \{u = v = 0\}$
  - But $q$ can be evaluated accurately and $p$ can't be
  - Why: dominant term of $q$ near $V(q)$ is
    $q_{dom} = (u^2 + v^2)(x^2 + y^2 + z^2)$ which is allowable
  - But $p_{dom} = (u^2 + v^2)(x + y + z)^2$ is not allowable
  - Idea of inductive decision procedure: look at all "dominant terms" near all components of $V(p)$
    * Ask if each dominant term can be evaluated accurately
    * Build accurate algorithm for $p$ by using accureate algorithms for each $p_{dom}$
    * Need Thm: $\exists$ accurate $p_{comp}$ iff $\forall p_{dom} \exists$ accurate $p_{dom,comp}$

# What are "dominant terms near $V(p)$"?

- Example:

$$\begin{aligned} p &= (x_1^8 + x_2^8) \cdot (x_3 + x_4 + x_5)^2 + (x_1^2 x_2^4 + x_1^4 x_2^2) \cdot ((x_3 - x_4)^4 + x_5^4) \\ &\equiv (x_1^8 + x_2^8) \cdot p_1 + (x_1^2 x_2^4 + x_1^4 x_2^2) \cdot p_2 \end{aligned}$$

- $V(p) = \{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = x_5 = 0\}$ allowable
- Near $\{x_1 = x_2 = 0\}$ dominant terms are
  1. $x_2^8 \cdot p_1$ when $|x_1| \ll x_2^2$
  2. $x_2^8 \cdot p_1 + x_1^2 x_2^4 \cdot p_2$ when $|x_1| \approx x_2^2$
  3. $x_1^2 x_2^4 \cdot p_2$ when $x_2^2 \ll |x_1| \ll |x_2|$
  4. $(x_1^2 x_2^4 + x_1^4 x_2^2) \cdot p_2$ when $|x_1| \approx |x_2|$
  5. $x_1^4 x_2^2 \cdot p_2$ when $x_1^2 \ll |x_2| \ll |x_1|$
  6. $x_1^4 x_2^2 \cdot p_2 + x_1^8 \cdot p_1$ when $x_1^2 \approx |x_2|$
  7. $x_1^8 \cdot p_1$ when $|x_2| \ll x_1^2$
- In Cases 1 and 7, $p_{dom}$ not allowable $\Rightarrow$ no accurate algorithm
- These cases arise from examining set of exponents of $x_1, x_2$, namely $(8,0)$, $(4,2)$, $(2,4)$, $(0,8)$: Newton polytope

## Proof Sketch that $V(p)$ allowable is necessary for accuracy (1/4):

- Def: Allow($x$) is the smallest allowable set containing $x$

$$\text{Allow}(x) = \mathrm{R}^n \cap (\cap_{i:\, x_i=0} Z_i) \cap (\cap_{i,j:\, x_i+x_j=0} S_{ij}) \cap (\cap_{i,j:\, x_i-x_j=0} D_{ij})$$

- Ex: Allow($(0, 1, -1, 2)$) $= Z_1 \cap S_{23}$
- If $V(p)$ not allowable, then

$$G(p) \equiv V(p) - \cup A$$

  is nonempty, where the union is over all allowable sets $A$ contained in $V(p)$
- Def: $G(p)$ called the set of points in "general position" in $V(p)$

# Proof Sketch (2/4)

- Assume no branching for simplicity
- Let $p_{comp}(x, \delta)$ denote result of computation.
- Main Lemma: Choose any $x$. One of following two cases must hold:
  1. $p_{comp}(x, \delta)$ is nonzero at $x$ for all $\delta$ in a Zariski-open set
  2. $p_{comp}(y, \delta) = 0$ for all $y \in \text{Allow}(x)$ and all $\delta$
- Suppose $V(p)$ not allowable. Choose any $x \in G(p) \subset V(p)$. Then either
  1. $p_{comp}(x, \delta)$ is nonzero at $x$ for all $\delta$ in a Zariski-open set
     but $p(x) = 0$, so the relative error is $\infty$
  2. $p_{comp}(y, \delta) = 0$ for all $y \in \text{Allow}(x)$ and all $\delta$
     but $p(y) \neq 0$ a.e., so the relative error is 1
- Can use continuity argument to show that relative error must be large on open set of $(x, \delta)$: i.e. large error on "large" set

# Proof Sketch (3/4)

- Main Lemma: Choose any $x$. One of following two cases must hold:
  1. $p_{comp}(x, \delta)$ is nonzero at $x$ for all $\delta$ in a Zariski-open set
  2. $p_{comp}(y, \delta) = 0$ for all $y \in \text{Allow}(x)$ and all $\delta$
- For simplicity, suppose no branching, no data reuse, nondeterminism
  - Implies that $p_{comp}(x, \delta)$ can be represented as a graph:
    * Source nodes representing data $x_i$, output edges connected to ...
    * Computational nodes, arranged in a tree, of following kinds:
      · 2-inputs, producing $fl(a \otimes b) = (a \otimes b)(1 + \delta_{\text{node}})$ ($\otimes \in \{+, -, \times\}$) with independent $|\delta_{\text{node}}| \le \epsilon$ for each node
      · 1-input, producing $fl(x \otimes x) = (x \otimes x)(1 + \delta_{\text{node}})$ (note: $fl(x - x) = 0$ exactly)
      · 1-input, producing $-x$ exactly
    * Destination node, one input, no output

# Proof Sketch (4/4)

- Main Lemma: Choose any $x$. One of following two cases must hold:
  1. $p_{comp}(x, \delta)$ is nonzero at $x$ for all $\delta$ in a Zariski-open set
  2. $p_{comp}(y, \delta) = 0$ for all $y \in \text{Allow}(x)$ and all $\delta$
- Def: Choose $x$. Call computational node "nontrivial" if it
  - Computes $fl(a \pm b)$, both $a$ and $b$ nonzero as polynomials in $\delta$
  - At least one of $a$ and $b$ not an input $x_i$
- Lemma: Output of all nontrivial nodes nonzero on Zariski-open set of $\delta$
- If ultimate output is from nontrivial node, done (Case 1)
- Otherwise, "trace back" zero output through tree as far as possible
- Can show (case analysis) that zero must result from one of
  - $x_i = 0$ (allowable)
  - $x_i \pm x_j = 0$ (allowable)
  - $x - x$ or $x + (-x)$ (in which case $alg(x, \delta) \equiv 0$)
- In any case, $p_{comp}(y, \delta)$ must be zero on Allow$(x)$ (Case 2)

# Other results and Future Work

- Need to complete decision procedure
- Want to incorporate
  - Determinism (simulate deterministic machine by nondeterministic one)
  - Constants (add $\{x : x_i \pm \alpha = 0\}$ to basic allowable sets for constant $\alpha$)
  - Domain $\mathcal{D}$ limited to (allowable?) semialgebraic sets
  - Division and rational functions
  - Other basic operation besides $\pm$, $\times$
    * How much more can we do with FMA $(x + y \cdot z)$, $x \cdot w - y \cdot z$, det$(3 \times 3)$,...
    * Use this to evaluate instruction sets, extended precision libraries
- Extend to interval arithmetic
- Perturbation theory
  - Conjecture: Accurate evaluation possible iff condition number can have certain simple singularities (depend on reciprocal distance to set of ill-posed problems)

## Conclusions

- **We have identified many classes of floating point expressions and matrix computations that permit**
  - **Accurate solutions: relative error $< 1$**
  - **Efficient solutions: time = poly(input size)**
- **Explored 3 natural models of arithmetic**
  - **Traditional Model (TM)**
  - **Long Exponent Model (LEM)**
  - **Short Exponent Model (SEM)**
- **New efficient algorithms for each: TM $\subsetneq$ LEM $\overset{\subseteq}{\neq ?}$ SEM**
- **New necessary condition for existence of accurate algorithm to evaluate $p(x)$ in TM – working on effective decision procedure**
- **Lots of open problems**
- **For more information see**
  - `www.cs.berkeley.edu/~demmel`
  - `math.mit.edu/~plamen`

# Extra Slides

# Improving LAPACK and ScaLAPACK

- Proposal by J. Demmel and J. Dongarra
- Many opportunities for improvment
  - Putting more of LAPACK into ScaLAPACK
  - Better (faster and/or more accurate) algorithms
  - New functions
  - Improving ease of use
  - Performance tuning
  - Support and reliability
- Seeking suggestions via on-line survey:
  - icl.cs.utk.edu/lapack-survey.html

# Better (faster and/or more accurate) algorithms

- **Offer 2 ”settings” for each driver:**
  1. **As fast as possible with ”standard” accuracy**
  2. **As accurate as possible with ”standard” speed**
  3. **What about memory usage?**
- **Consider SVD/EVD: Choice of algorithm depends on**
  - **values only / few vectors / many vectors,**
  - **left vectors / right / both**
- **Options for ”fast as possible”**
  - **Successive Band Reduction (Lang et al)**
  - **Howell/Fulton bidiagonalization**
  - **One-sided bidiagonalization (Ralha et al, Barlow et al, ...)**
  - **Other?**
- **Options for ”as accurate as possible”**
  - **Jacobi SVD (Drmač et al)**
  - **Symmetric EVD?**