CHECKMATE!

A Brief Introduction to Game Theory

The World

Kasparov

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Welcome!

- Introduction
- Topic motivation, goals
- Talk overview
    - ◊ Combinatorial game theory basics w/examples
    - ◊ "Computational" game theory
    - ◊ Analysis of some simple games
    - ◊ Research highlights

Game Theory: Economic or Combinatorial?

- Economic
  - ◊ von Neumann and Morgenstern's 1944 *Theory of Games and Economic Behavior*
  - ◊ Matrix games
  - ◊ Prisoner's dilemma
  - ◊ Incomplete info, simultaneous moves
  - ◊ Goal: Maximize payoff
- Combinatorial
  - ◊ Sprague and Grundy's 1939 *Mathematics and Games*
  - ◊ Board (table) games
  - ◊ Nim, Domineering
  - ◊ Complete info, alternating moves
  - ◊ Goal: Last move

Why study games?

- Systems design
  - ◊ Decomposition into parts with limited interactions
- Complexity Theory
- Management
  - ◊ Determine area to focus energy / resources
- Artificial Intelligence testing grounds
- “People want to understand the things that people like to do, and people like to play games” – Berlekamp & Wolfe

Combinatorial Game Theory History

- Early Play
  - ◊ Egyptian wall painting of Senat (c. 3000 BC)
- Theory
  - ◊ C. L. Bouton's analysis of Nim [1902]
  - ◊ Sprague [1936] and Grundy [1939] Impartial games and Nim
  - ◊ Knuth *Surreal Numbers* [1974]
  - ◊ Conway *On Numbers and Games* [1976]
  - ◊ Prof. Elwyn Berlekamp (UCB), Conway, & Guy *Winning Ways* [1982]

What is a combinatorial game?

- Two players (Left & Right) move alternately
- No chance, such as dice or shuffled cards
- Both players have perfect information
  - ◊ No hidden information, as in Stratego & Magic
- The game is finite – it must eventually end
- There are no draws or ties
- Normal Play: Last to move wins!

What games are out, what are in?

- Out
  - ◊ All card games
  - ◊ All dice games

- In
  - ◊ Nim, Domineering, Dots-and-Boxes, Go, etc.
- In, but not normal play
  - ◊ Chess, Checkers, Othello, Tic-Tac-Toe, etc.

Combinatorial Game Theory
The Big Picture

- Whose turn is not part of the game
- SUMS of games
  - ◊ You play games $G_1 + G_2 + G_3 + \ldots$
  - ◊ You decide which game is most important
  - ◊ You want the last move (in normal play)
  - ◊ Analogy: Eating with a friend, want the last bite

Classification of Games

- Impartial
  - ◊ Same moves available to each player
  - ◊ Example: Nim
- Partisan
  - ◊ The two players have different options
  - ◊ Example: Domineering

Nim : <u>The</u> Impartial Game pt. I

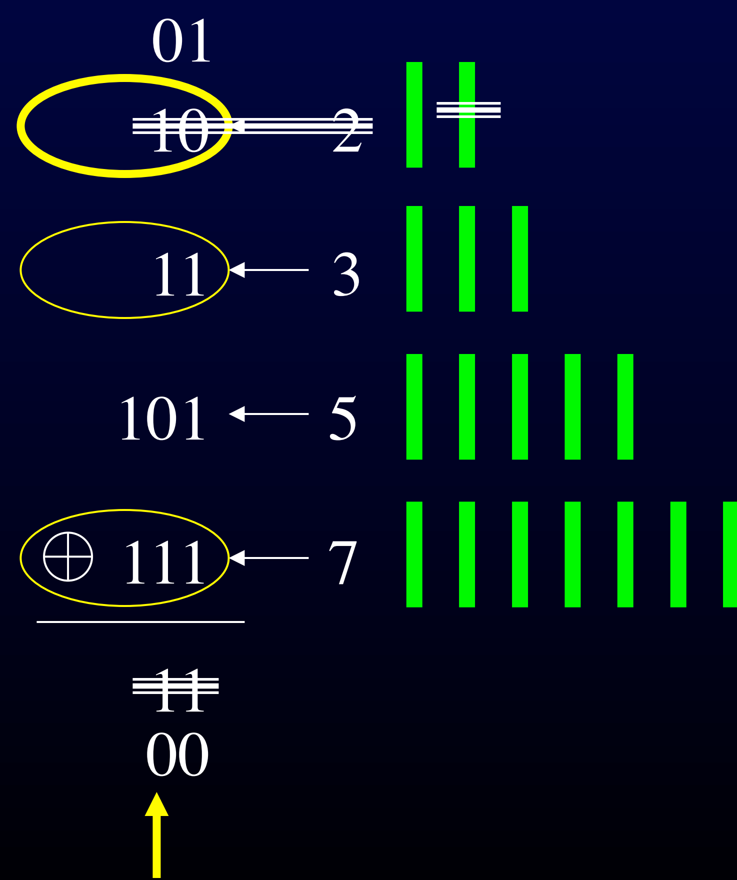
- Rules:
  - ◊ Several heaps of beans
  - ◊ On your turn, select a heap, and remove any positive number of beans from it, maybe all
- Goal
  - ◊ Take the last bean
- Example w/4 piles: (2,3,5,7)

Nim: <u>The</u> Impartial Game pt. II

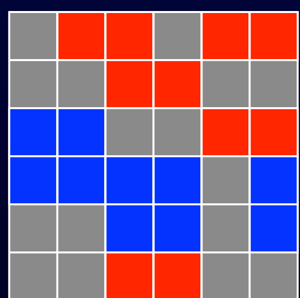
- Dan plays room in (2,3,5,7) Nim
- Pair up, play (2,3,5,7)
  - ◊ Query:
    - First player win or lose?
    - Perfect strategy?
  - ◊ Feedback, theories?
- Every impartial game is equivalent to a (bogus) Nim heap

Nim: The Impartial Game pt. III

- Winning or losing?
  - ◊ Binary rep. of heaps
  - ◊ Nim Sum == XOR ⊕
  - ◊ Zero == Losing, 2nd P win
- Winning move?
  - ◊ Find MSB in Nim Sum
  - ◊ Find heap w/1 in that place
  - ◊ Invert all heap's bits from sum to make sum zero



Domineering: A partisan game

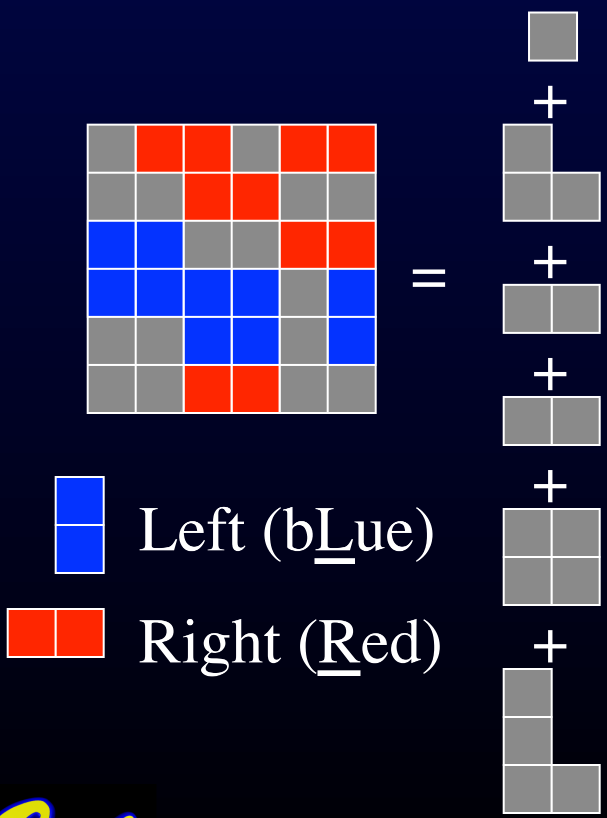


Left (bLue)

Right (Red)

- Rules (on your turn):
  - ◊ Place a domino on the board
  -  ◊ Left places them North-South
  -  ◊ Right places them East-West
- Goal
  - ◊ Place the last domino
- Example game
- Query: Who wins here?

Domineering: A partisan game



- Key concepts
  - ◊ By moving correctly, you guarantee yourself future moves.
  - ◊ For many positions, you want to move, since you can steal moves. This is a "hot" game.
  - ◊ This game decomposes into non-interacting parts, which we separately analyze and bring results together.

What do we want to know about a particular game?

- What is the value of the game?
  - ◊ Who is ahead and by how much?
  - ◊ How big is the next move?
  - ◊ Does it matter who goes first?
- What is a winning / drawing strategy?
  - ◊ To know a game's value and winning strategy is to have solved the game
  - ◊ Can we easily summarize strategy?

Combinatorial Game Theory The Basics I - Game definition

- A game, G, between two players, Left and Right, is defined as a pair of sets of games:
  - ◊ $G = \{G^L \mid G^R\}$
  - ◊ G^L is the typical Left option (i.e., a position Left can move to), similarly for Right.
  - ◊ G^L need not have a unique value
  - ◊ Thus if $G = \{a, b, c, \ldots \mid d, e, f, \ldots\}$, G^L means *a* or *b* or *c* or … and G^R means *d* or *e* or *f* or ...

Combinatorial Game Theory
The Basics II - Examples: 0

- The simplest game, the Endgame, born day 0
  - ◊ Neither player has a move, the game is over
  - ◊ { Ø | Ø } = { | }, we denote by 0 (a number!)
  - ◊ Example of *P*, previous/second-player win, losing
  - ◊ Examples from games we've seen:

Nim	Domineering	Game Tree
		

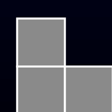
Combinatorial Game Theory
The Basics II - Examples: *

- The next simplest game, * (“Star”), born day 1
  - ◊ First player to move wins
  - ◊ { 0 | 0 } = *, this game is not a number, it’s <u>fuzzy</u>!
  - ◊ Example of *N*, a next/first-player win, winning
  - ◊ Examples from games we’ve seen:

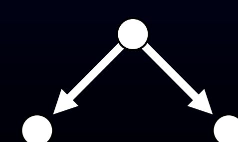
<u>Nim</u>

1 

<u>Domineering</u>



<u>Game Tree</u>



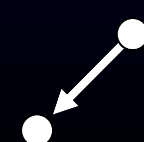
Combinatorial Game Theory
The Basics II - Examples: 1

- Another simple game, 1, born day 1
  - ◊ Left wins no matter who starts
  - ◊ { 0 | } = 1, this game is a number
  - ◊ Called a Left win. Partisan games only.
  - ◊ Examples from games we've seen:

Nim

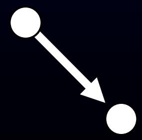
Domineering

Game Tree



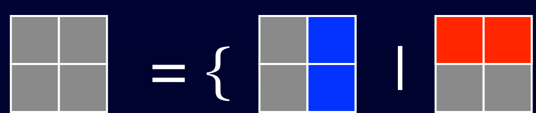
Combinatorial Game Theory
The Basics II - Examples: –1

- Similarly, a game, –1, born day 1
  - ◊ Right wins no matter who starts
  - ◊ { | 0 } = –1, this game is a number.
  - ◊ Called a Right win. Partisan games only.
  - ◊ Examples from games we've seen:

Nim	Domineering	Game Tree
		

Combinatorial Game Theory
The Basics II - Examples

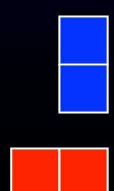
- Calculate value for Domineering game G:

G =  = { | }

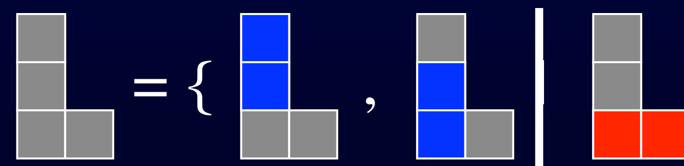
= { 1 | – 1 }

= ± 1

…this is a fuzzy hot value, confused with 0. 1st player wins.



- Calculate value for Domineering game G:

G =  = { , | }

= { – 1 , 0 | 1 }

= { 0 | 1 }

= { .5 }

…this is a cold fractional value. Left wins regardless who starts.

Combinatorial Game Theory
The Basics III - Outcome classes

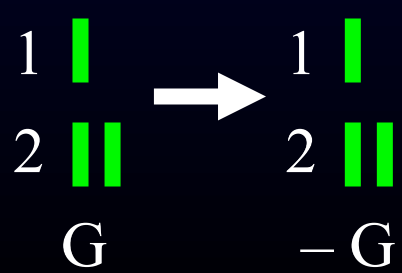
- With normal play, every game belongs to one of four outcome classes (compared to 0):
  - ◊ Zero (=)
  - ◊ Negative (<)
  - ◊ Positive (>)
  - ◊ Fuzzy (||), incomparable, confused

Left starts	Right starts: and L has winning strategy	Right starts: and R has winning strategy
and R has winning strategy	ZERO $G = 0$ 2nd wins	NEGATIVE $G < 0$ R wins
and L has winning strategy	POSITIVE $G > 0$ L wins	FUZZY $G \parallel 0$ 1st wins

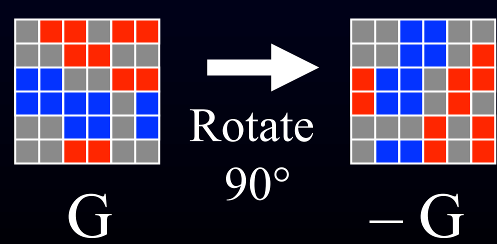
Combinatorial Game Theory
The Basics IV - Negatives & Sums

- Negative of a game: definition
  - ◊ $-G = \{-G^R \mid -G^L\}$
  - ◊ Similar to switching places with your opponent
  - ◊ Impartial games are their own neg., so $-G = G$
  - ◊ Examples from games we've seen:

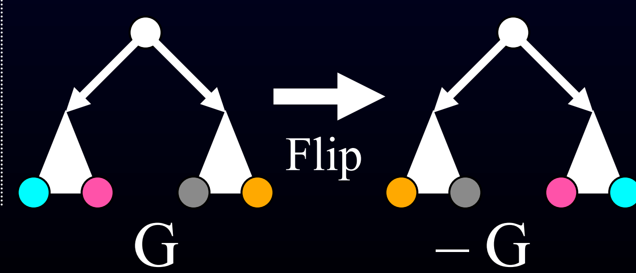
Nim



Domineering



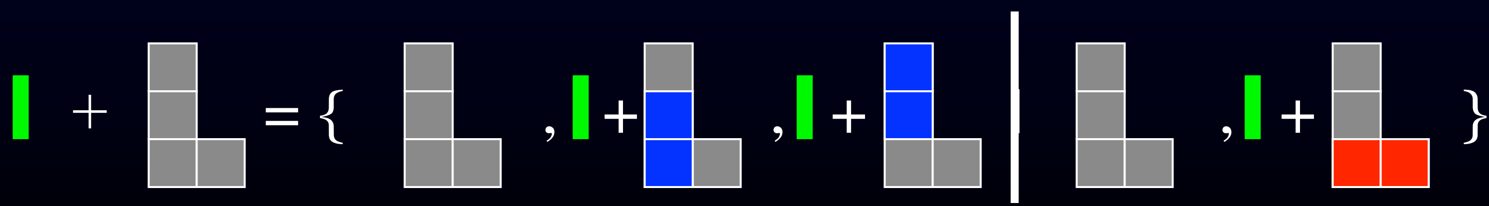
Game Tree



Combinatorial Game Theory
The Basics IV - Negatives & Sums

- Sums of games: definition
  - ◊ $G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$
  - ◊ The player whose turn it is selects one component and makes a move in it.
  - ◊ Examples from games we've seen:

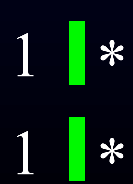
$$G \;+\; H \;=\; \{\, G^L + H,\ G + H_1^L,\ G + H_2^L \mid G^R + H,\ G + H^R \,\}$$



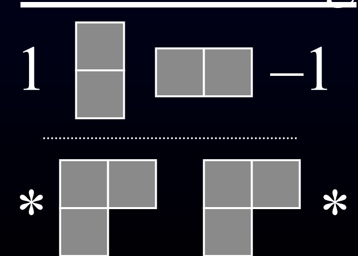
Combinatorial Game Theory
The Basics IV - Negatives & Sums

- $G + 0 = G$
  - ◊ The Endgame doesn't change a game's value
- $G + (-G) = 0$
  - ◊ "= 0" means is a zero game, 2nd player can win
  - ◊ Examples: $1 + (-1) = 0$ and $* + * = 0$

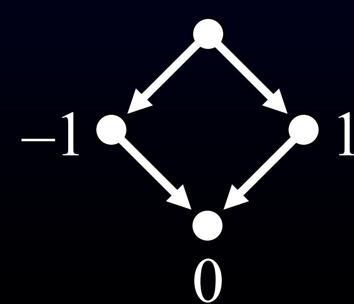
Nim



Domineering



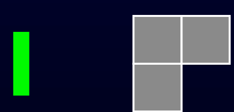
Game Tree



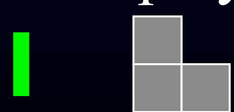
Combinatorial Game Theory
The Basics IV - Negatives & Sums

- G = H
  - ◊ If the game G + (–H) = 0, i.e., a 2nd player win
  - ◊ Examples from games we've seen:

Is G = H ?

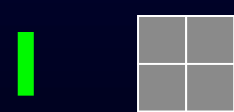


Play G + (–H) and
see if 2nd player win

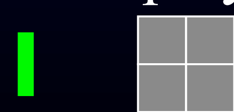


Yes!

Is G = H ?



Play G + (–H) and
see if 2nd player win



No...

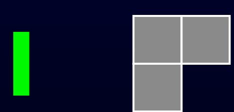
Left

Right

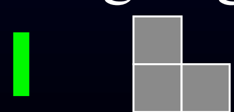
Combinatorial Game Theory
The Basics IV - Negatives & Sums

- $G \geq H$ (Games form a partially ordered set!)
  - ◊ If Left can win the sum $G + (-H)$ going 2nd
  - ◊ Examples from games we've seen:

Is $G \geq H$?

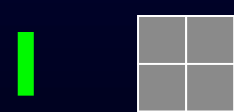


Play $G + (-H)$ and see if Left wins going 2nd

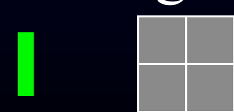


Yes!

Is $G \geq H$?



Play $G + (-H)$ and see if Left wins going 2nd



No...

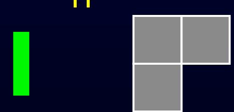

Left


Right

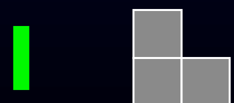
Combinatorial Game Theory
The Basics IV - Negatives & Sums

- G || H   (G is incomparable with H)
  - ◊ If G + (–H) is || with 0, i.e., a 1st player win
  - ◊ Examples from games we've seen:

Is G || H ?

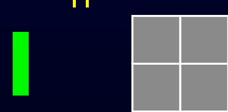


Play G + (–H) and see if 1st player win

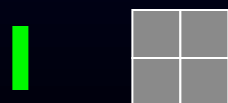


No...

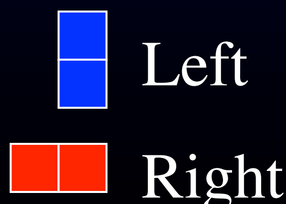
Is G || H ?



Play G + (–H) and see if 1st player win

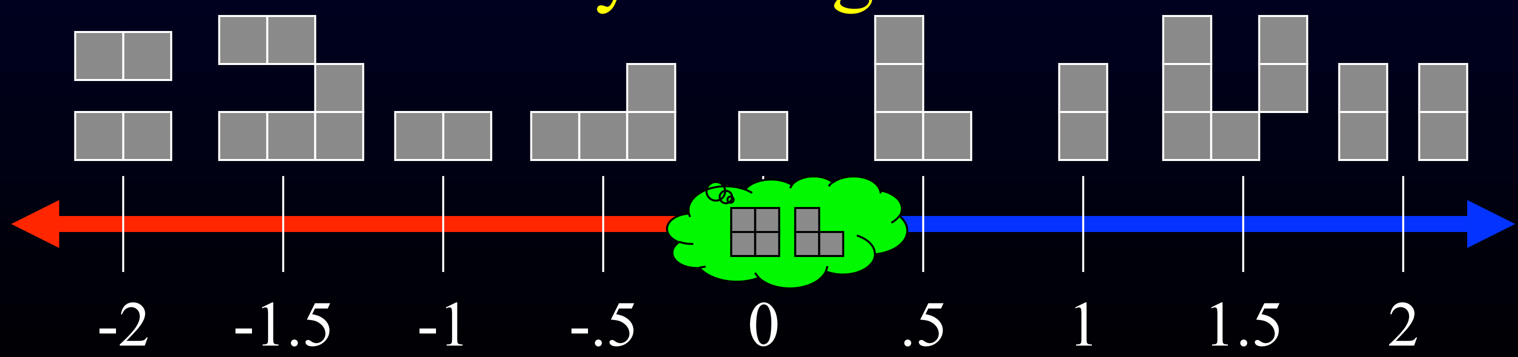


YES!



Combinatorial Game Theory
The Basics IV - Values of games

- What is the value of a fuzzy game?
  - ◊ It's neither > 0, < 0 nor = 0, but confused with 0
  - ◊ Its place on the number scale is indeterminate
  - ◊ Often represented as a "cloud"
- Let's tie the theory all together!



Combinatorial Game Theory
The Basics V - Final thoughts

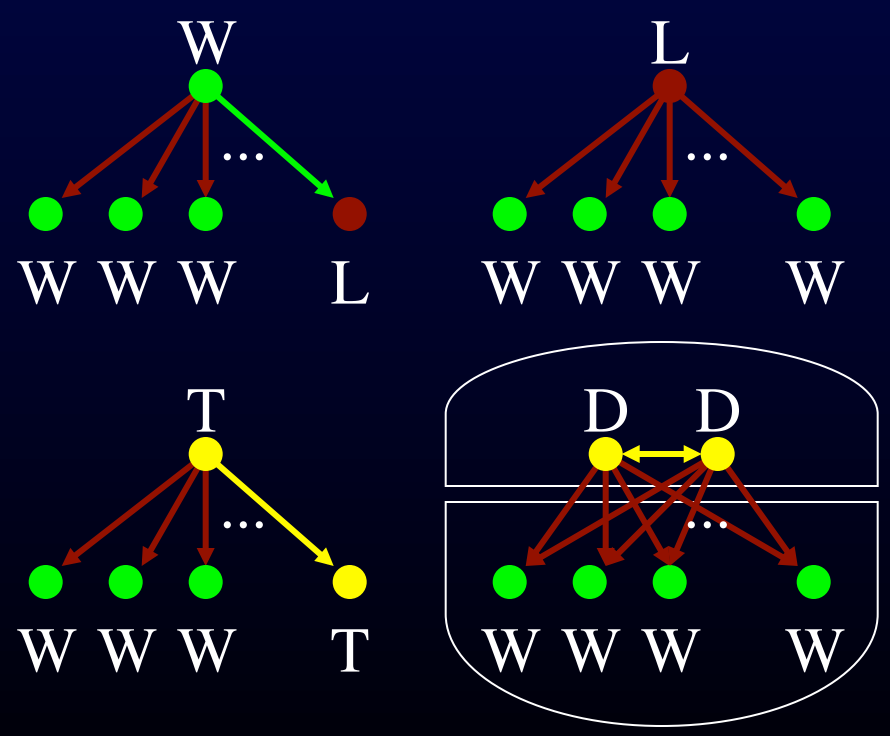
- There's much more!
  - ◊ More values
    - Up, Down, Tiny, etc.
  - ◊ Simplicity, Mex rule
  - ◊ Dominating options
  - ◊ Reversible moves
  - ◊ Number avoidance
  - ◊ Temperatures
- Normal form games
  - ◊ Last to move wins, no ties
  - ◊ Whose turn not in game
  - ◊ Rich mathematics
  - ◊ Key: Sums of games
  - ◊ Many (most?) games are not normal form!
    - What do we do then?

“Computational” Game Theory (for non-normal play games)

- Large games
  - ◊ Can theorize strategies, build AI systems to play
  - ◊ Can study endgames, smaller version of original
    - Examples: Quick Chess, 9x9 Go, 6x6 Checkers, etc.
- Small-to-medium games
  - ◊ Can have computer solve and <u>teach us strategy</u>
  - ◊ GAMESMAN does exactly this

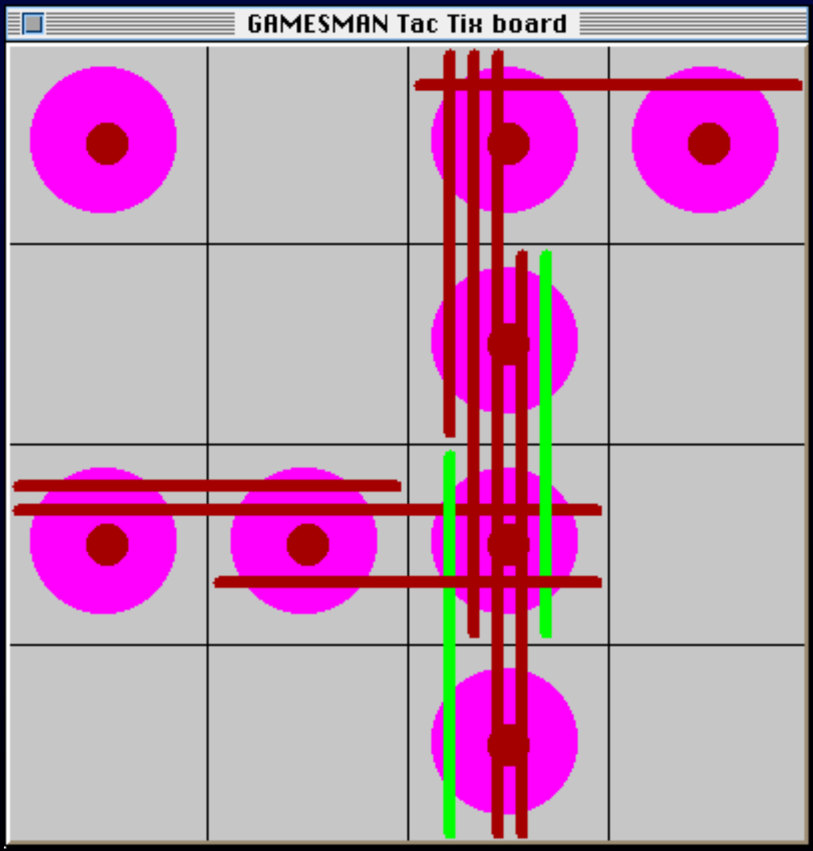
Computational Game Theory

- Simplify games / value
  - ◊ Store turn in position
  - ◊ Each position is (for player whose turn it is)
    - <u>Winning</u> (∃ losing child)
    - <u>Losing</u> (All children winning)
    - <u>Tieing</u> (!∃ losing child, but ∃ tieing child)
    - <u>Drawing</u> (can't force a win or be forced to lose)



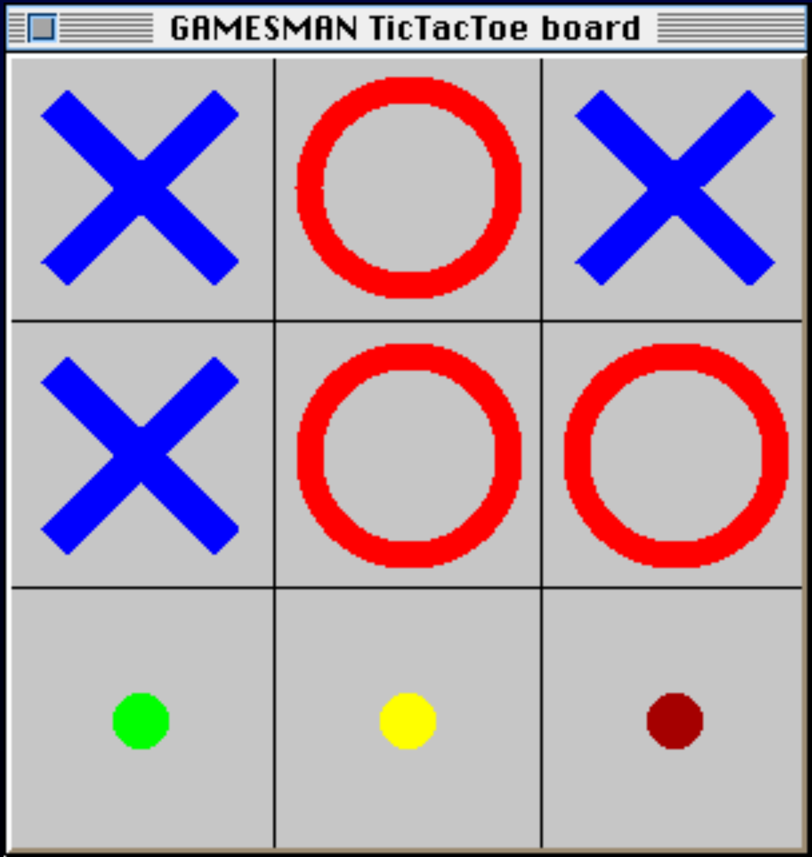
GAMESMAN
Analysis: TacTix, or 2-D Nim

- Rules (on your turn):
  - ◊ Take as many pieces as you want from any contiguous row / column
- Goal
  - ◊ Take the last piece
- Query
  - ◊ Column = Nim heap?
  - ◊ Zero shapes



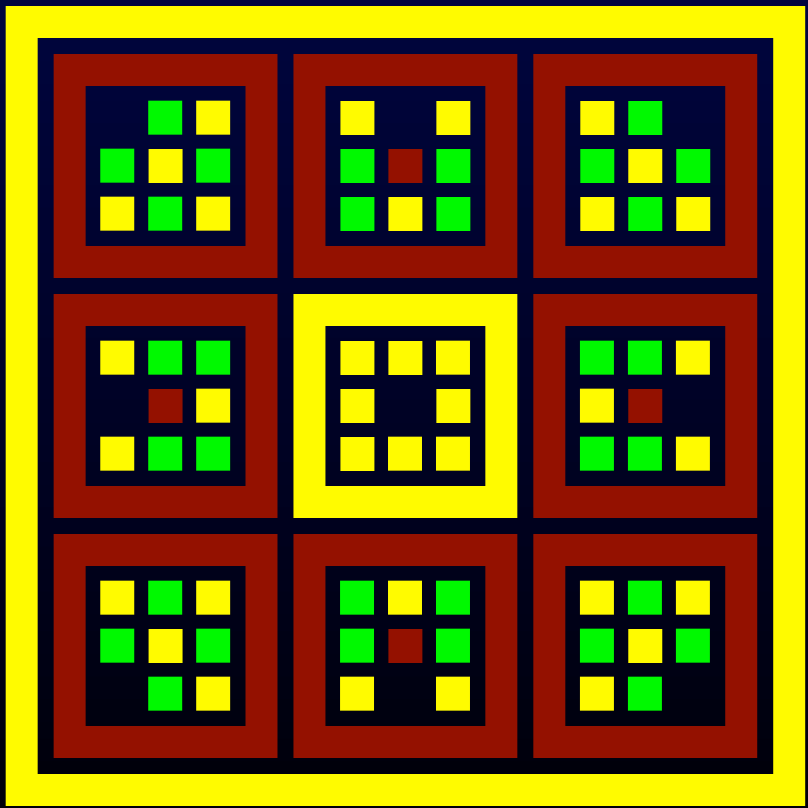
GAMESMAN
Analysis: Tic-Tac-Toe

- Rules (on your turn):
  - ◊ Place your X or O in an empty slot
- Goal
  - ◊ Get 3-in-a-row first in any row/column/diag.
- Misére is tricky



GAMESMAN
Tic-Tac-Toe Visualization

- Visualization of values
- Example with Misére
  - ◊ Outer rim is position
  - ◊ Next levels are values of <u>moves</u> to that position
  - ◊ Recursive image
  - ◊ Legend: Lose, Tie, Win

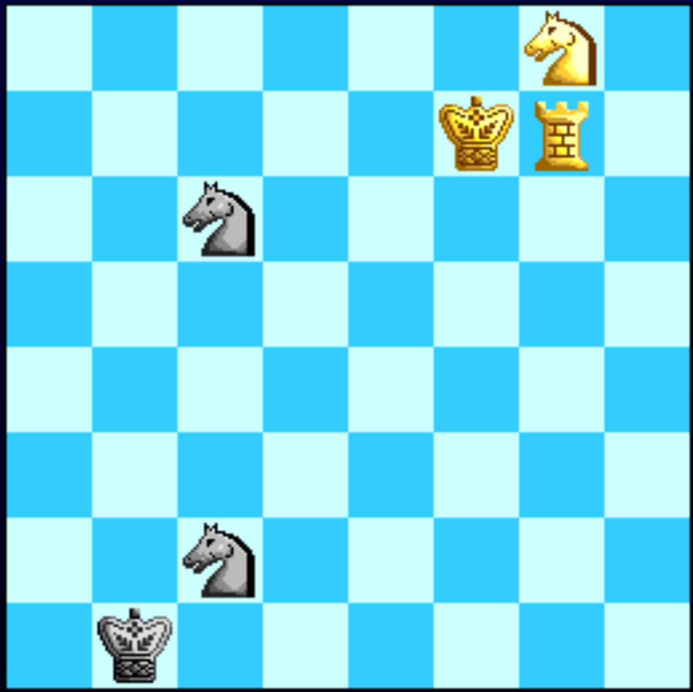


Exciting Game Theory Research at Berkeley

- Combinatorial Game Theory Workshop
  - ◊ MSRI July 24-28th, 2000
  - ◊ 1994 Workshop book: Games of No Chance
- Prof. Elwyn Berlekamp
  - ◊ Dots & Boxes, Go endgames
  - ◊ Economist's View of Combinatorial Games

Exciting Game Theory Research
Chess

- Kasparov vs.
  - ◊ World, Deep Blue II
- Endgames, tablebases
  - ◊ Stiller, Nalimov
  - ◊ Combinatorial GT applied
    - Values found [Elkies, 1996]
  - ◊ SETI@Home parallel power to build database?
  - ◊ Historical analysis...



White to move, wins in move 243 with Rd7xNe7

Exciting Game Theory Research
Solving games

- 4x4x4 Tic-Tac-Toe [Patashnik, 1980]
- Connect-4 [Allen, 1989; Allis, 1988]
- Go-Moku [Allis et al., 1993]
- Nine Men's Morris [Gasser, 1996]
  - ◊ One of oldest games – boards found c. 1400 BC
- Checkers almost solved [Schaeffer, 1996]

Summary

- Combinatorial game theory, learned games
- Computational game theory, GAMESMAN
- Reviewed research highlights