CHECKMATE!

A Brief Introduction to Game Theory

The World

Kasparov

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UC Berkeley
Welcome!

• Introduction
• Topic motivation, goals
• Talk overview
  ◊ Combinatorial game theory basics w/examples
  ◊ “Computational” game theory
  ◊ Analysis of some simple games
  ◊ Research highlights
Game Theory: Economic or Combinatorial?

• Economic
  ◊ von Neumann and Morgenstern’s 1944 *Theory of Games and Economic Behavior*
  ◊ Matrix games
  ◊ Prisoner’s dilemma
  ◊ Incomplete info, simultaneous moves
  ◊ Goal: Maximize payoff

• Combinatorial
  ◊ Sprague and Grundy’s 1939 *Mathematics and Games*
  ◊ Board (table) games
  ◊ Nim, Domineering
  ◊ Complete info, alternating moves
  ◊ Goal: Last move
Why study games?

• Systems design
  ◊ Decomposition into parts with limited interactions
• Complexity Theory
• Management
  ◊ Determine area to focus energy / resources

• Artificial Intelligence testing grounds
• “People want to understand the things that people like to do, and people like to play games” – Berlekamp & Wolfe
Combinatorial Game Theory

History

• Early Play
  ◊ Egyptian wall painting of Senat (c. 3000 BC)

• Theory
  ◊ C. L. Bouton’s analysis of Nim [1902]
  ◊ Sprague [1936] and Grundy [1939] Impartial games and Nim

◊ Knuth *Surreal Numbers* [1974]
◊ Conway *On Numbers and Games* [1976]
◊ Prof. Elwyn Berlekamp (UCB), Conway, & Guy *Winning Ways* [1982]
What is a combinatorial game?

- Two players (Left & Right) move alternately
- No chance, such as dice or shuffled cards
- Both players have perfect information
  - No hidden information, as in Stratego & Magic
- The game is finite – it must eventually end
- There are no draws or ties
- Normal Play: Last to move wins!
What games are out, what are in?

- **Out**
  - ♦ All card games
  - ♦ All dice games
- **In**
  - ♦ Nim, Domineering, Dots-and-Boxes, Go, etc.
- **In, but not normal play**
  - ♦ Chess, Checkers, Othello, Tic-Tac-Toe, etc.
Combinatorial Game Theory
The Big Picture

- Whose turn is not part of the game
- SUMS of games
  - You play games $G_1 + G_2 + G_3 + \ldots$
  - You decide which game is most important
  - You want the last move (in normal play)
  - Analogy: Eating with a friend, want the last bite
Classification of Games

- **Impartial**
  - Same moves available to each player
  - Example: Nim

- **Partisan**
  - The two players have different options
  - Example: Domineering
Nim: The Impartial Game pt. I

• Rules:
  ◊ Several heaps of beans
  ◊ On your turn, select a heap, and remove any positive number of beans from it, maybe all

• Goal
  ◊ Take the last bean

• Example w/4 piles: (2,3,5,7)
Nim: **The Impartial Game** pt. II

- Dan plays room in (2,3,5,7) Nim
- Pair up, play (2,3,5,7)
  - Query:
    - First player win or lose?
    - Perfect strategy?
  - Feedback, theories?
- Every impartial game is equivalent to a (bogus) Nim heap
Nim: The Impartial Game pt. III

• Winning or losing?
  ◊ Binary rep. of heaps
  ◊ Nim Sum == XOR ⊕
  ◊ Zero == Losing, 2nd P win

• Winning move?
  ◊ Find MSB in Nim Sum
  ◊ Find heap w/1 in that place
  ◊ Invert all heap’s bits from sum to make sum zero
Domineering: A partisan game

- Rules (on your turn):
  - Place a domino on the board
  - Left places them North-South
  - Right places them East-West

- Goal
  - Place the last domino

- Example game

- Query: Who wins here?
Domineering: A partisan game

- Key concepts
  - By moving correctly, you guarantee yourself future moves.
  - For many positions, you want to move, since you can steal moves. This is a “hot” game.
  - This game decomposes into non-interacting parts, which we separately analyze and bring results together.

Left (bLue) = Right (Red)
What do we want to know about a particular game?

- What is the value of the game?
  - Who is ahead and by how much?
  - How big is the next move?
  - Does it matter who goes first?

- What is a winning / drawing strategy?
  - To know a game’s value and winning strategy is to have solved the game
  - Can we easily summarize strategy?
Combinatorial Game Theory
The Basics I - Game definition

• A game, G, between two players, Left and Right, is defined as a pair of sets of games:
  ◊ \( G = \{G^L \; | \; G^R \} \)
  ◊ \( G^L \) is the typical Left option (i.e., a position Left can move to), similarly for Right.
  ◊ \( G^L \) need not have a unique value
  ◊ Thus if \( G = \{a, b, c, \ldots \; | \; d, e, f, \ldots \} \), \( G^L \) means \( a \) or \( b \) or \( c \) or \( \ldots \) and \( G^R \) means \( d \) or \( e \) or \( f \) or \( \ldots \)
Combinatorial Game Theory
The Basics II - Examples: 0

- The simplest game, the Endgame, born day 0
  ◊ Neither player has a move, the game is over
  ◊ \( \{ \emptyset | \emptyset \} = \{ | \} \), we denote by 0 (a number!)
  ◊ Example of \( P \), previous/second-player win, losing
  ◊ Examples from games we’ve seen:
    - Nim
    - Domineering
    - Game Tree

A Brief Introduction to Game Theory
The next simplest game, * ("Star"), born day 1

◊ First player to move wins

◊ \{ 0 \mid 0 \} = *, this game is not a number, it’s fuzzy!

◊ Example of \( N \), a next/first-player win, winning

◊ Examples from games we’ve seen:

<table>
<thead>
<tr>
<th>Nim</th>
<th>Domineering</th>
<th>Game Tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Combinatorial Game Theory
The Basics II - Examples: 1

• Another simple game, 1, born day 1
  ◊ Left wins no matter who starts
  ◊ \( \{ 0 \mid \} = 1 \), this game is a number
  ◊ Called a Left win. Partisan games only.

◊ Examples from games we’ve seen:

<table>
<thead>
<tr>
<th>Nim</th>
<th>Domineering</th>
<th>Game Tree</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Nim" /></td>
<td><img src="image" alt="Domineering" /></td>
<td><img src="image" alt="Game Tree" /></td>
</tr>
</tbody>
</table>
Combinatorial Game Theory
The Basics II - Examples: $-1$

• Similarly, a game, $-1$, born day 1
  ◊ Right wins no matter who starts
  ◊ $\{ \mid 0 \} = -1$, this game is a number.
  ◊ Called a Right win. Partisan games only.
  ◊ Examples from games we’ve seen:

<table>
<thead>
<tr>
<th>Nim</th>
<th>Domineering</th>
<th>Game Tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Image](36x36 to 122x113)</td>
<td><img src="142x492" alt="Domineering" /></td>
<td><img src="288x263" alt="Game Tree" /></td>
</tr>
</tbody>
</table>

A Brief Introduction to Game Theory
Combinatorial Game Theory
The Basics II - Examples

- Calculate value for Domineering game $G$:
  $$G = \begin{array}{c|c}
    \text{Left} & \text{Right} \\
    \hline
    \begin{array}{c}
    \text{Blue} \\
    \text{Red}
    \end{array} & \begin{array}{c}
    \text{Gray}
    \end{array}
  \end{array}$$

  $$= \{ 1 | -1 \}$$

  $$= \pm 1$$

  …this is a fuzzy hot value, confused with $0$. 1st player wins.

- Calculate value for Domineering game $G$:
  $$G = \begin{array}{c|c}
    \text{Left} & \text{Right} \\
    \hline
    \begin{array}{c}
    \text{Blue} \\
    \text{Gray}
    \end{array} & \begin{array}{c}
    \text{Gray}
    \end{array}
  \end{array}$$

  $$= \{ -1, 0 | 1 \}$$

  $$= \{ 0 | 1 \}$$

  $$= \{ .5 \}$$

  …this is a cold fractional value. Left wins regardless who starts.
Combinatorial Game Theory
The Basics III - Outcome classes

- With **normal play**, every game belongs to one of four outcome classes (compared to 0):
  - **Zero (=)**
  - **Negative (<)**
  - **Positive (>)**
  - **Fuzzy (||)**, incomparable, confused

<table>
<thead>
<tr>
<th>Outcome Class</th>
<th>Description</th>
<th>Winner</th>
<th>Winning Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero (=)</td>
<td>G = 0</td>
<td>2nd</td>
<td>and L has winning strategy</td>
</tr>
<tr>
<td>Negative (&lt;)</td>
<td>G &lt; 0</td>
<td>R</td>
<td>and R has winning strategy</td>
</tr>
<tr>
<td>Positive (&gt;)</td>
<td>G &gt; 0</td>
<td>L</td>
<td>and L has winning strategy</td>
</tr>
<tr>
<td>Fuzzy (</td>
<td></td>
<td>)</td>
<td>G</td>
</tr>
</tbody>
</table>

Left starts

Right starts
The Basics IV - Negatives & Sums

- Negative of a game: definition
  \[ -G = \{-G^R | -G^L\} \]
  - Similar to switching places with your opponent
  - Impartial games are their own neg., so \(-G = G\)
  - Examples from games we’ve seen:

<table>
<thead>
<tr>
<th>Game</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nim</td>
<td><img src="image" alt="Nim Game" /></td>
</tr>
<tr>
<td>Domineering</td>
<td><img src="image" alt="Domineering Game" /></td>
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<td>Game Tree</td>
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</table>
Combinatorial Game Theory
The Basics IV - Negatives & Sums

• Sums of games: definition

◊ \( G + H = \{ G^L + H, G + H^L \mid G^R + H, G + H^R \} \)

◊ The player whose turn it is selects one component and makes a move in it.

◊ Examples from games we’ve seen:

\[
G + H = \{ G^L + H, G + H_1^L, G + H_2^L \mid G^R + H, G + H^R \}
\]

\[
\begin{align*}
\text{I} + & \begin{array}{c}
\text{L} \\
\text{L}
\end{array} = \{ \begin{array}{c}
\text{L} \\
\text{L}
\end{array}, \text{I} + \begin{array}{c}
\text{L}
\end{array}, \text{I} + \begin{array}{c}
\text{L}
\end{array} \mid \begin{array}{c}
\text{L}
\end{array}, \text{I} + \begin{array}{c}
\text{L}
\end{array} \}
\end{align*}
\]
Combinatorial Game Theory
The Basics IV - Negatives & Sums

- \( G + 0 = G \)
  - \( \diamond \) The Endgame doesn’t change a game’s value

- \( G + (–G) = 0 \)
  - \( \diamond \) “= 0” means is a zero game, 2nd player can win
  - \( \diamond \) Examples: \( 1 + (–1) = 0 \) and \( \ast + \ast = 0 \)

\[ \begin{array}{c|c|c}
\text{Nim} & \text{Domineering} & \text{Game Tree} \\
1 \mid \ast & 1 \mid \ast \ast -1 & \\
1 \mid \ast & \ast \ast -1 & \end{array} \]

A Brief Introduction to Game Theory
Combinatorial Game Theory
The Basics IV - Negatives & Sums

- $G = H$
  
  ◊ If the game $G + (-H) = 0$, i.e., a 2nd player win
  ◊ Examples from games we’ve seen:

  Is $G = H$?

  Play $G + (-H)$ and see if 2nd player win

  Yes!

  Is $G = H$?

  Play $G + (-H)$ and see if 2nd player win

  No...
Combinatorial Game Theory
The Basics IV - Negatives & Sums

- $G \geq H$ (Games form a partially ordered set!)
  
 ◊ If Left can win the sum $G + (-H)$ going 2nd
  
 ◊ Examples from games we’ve seen:

<table>
<thead>
<tr>
<th>Is $G \geq H$?</th>
<th>Is $G \geq H$?</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Game 1" /></td>
<td><img src="image2.png" alt="Game 2" /></td>
</tr>
<tr>
<td>Play $G + (-H)$ and see if Left wins going 2nd</td>
<td>Play $G + (-H)$ and see if Left wins going 2nd</td>
</tr>
<tr>
<td>Yes!</td>
<td>No...</td>
</tr>
</tbody>
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Combinatorial Game Theory
The Basics IV - Negatives & Sums

• G || H (G is incomparable with H)

◊ If G + (–H) is || with 0, i.e., a 1st player win

◊ Examples from games we’ve seen:

Is G || H ?

<p>| | |</p>
<table>
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Play G + (–H) and see if 1st player win

No...

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Is G || H ?

<p>| | |</p>
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</table>

Play G + (–H) and see if 1st player win

Left

Right

YES!
Combinatorial Game Theory
The Basics IV - Values of games

• What is the value of a fuzzy game?
  ◊ It’s neither > 0, < 0 nor = 0, but confused with 0
  ◊ Its place on the number scale is indeterminate
  ◊ Often represented as a “cloud”

• Let’s tie the theory all together!

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A Brief Introduction to Game Theory
Combinatorial Game Theory
The Basics V - Final thoughts

• There’s much more!
  ◊ More values
    • Up, Down, Tiny, etc.
  ◊ Simplicity, Mex rule
  ◊ Dominating options
  ◊ Reversible moves
  ◊ Number avoidance
  ◊ Temperatures

• Normal form games
  ◊ Last to move wins, no ties
  ◊ Whose turn not in game
  ◊ Rich mathematics
  ◊ Key: Sums of games
  ◊ Many (most?) games are not normal form!
    • What do we do then?
“Computational” Game Theory (for non-normal play games)

- **Large games**
  - Can theorize strategies, build AI systems to play
  - Can study endgames, smaller version of original
    - Examples: Quick Chess, 9x9 Go, 6x6 Checkers, etc.

- **Small-to-medium games**
  - Can have computer solve and teach us strategy
  - GAMESMAN does exactly this
Computational Game Theory

- **Simplify games / value**
  - Store turn in position
  - Each position is (for player whose turn it is)
    - Winning ($\exists$ losing child)
    - Losing (All children winning)
    - Tieing ($\nexists$ losing child, but $\exists$ tieing child)
    - Drawing (can’t force a win or be forced to lose)
GAMESMAN
Analysis: TacTix, or 2-D Nim

• Rules (on your turn):
  ◊ Take as many pieces as you want from any contiguous row / column

• Goal
  ◊ Take the last piece

• Query
  ◊ Column = Nim heap?
  ◊ Zero shapes
GAMESMAN
Analysis: Tic-Tac-Toe

• Rules (on your turn):
  ♦ Place your X or O in an empty slot
• Goal
  ♦ Get 3-in-a-row first in any row/column/diag.
• Misère is tricky
GAMESMAN
Tic-Tac-Toe Visualization

- Visualization of values
- Example with Misère
  ◇ Outer rim is position
  ◇ Next levels are values of moves to that position
  ◇ Recursive image
  ◇ Legend: Lose, Tie, Win
Exciting Game Theory Research at Berkeley

• Combinatorial Game Theory Workshop
  ◊ MSRI July 24-28th, 2000
  ◊ 1994 Workshop book: Games of No Chance

• Prof. Elwyn Berlekamp
  ◊ Dots & Boxes, Go endgames
  ◊ Economist’s View of Combinatorial Games
Exciting Game Theory Research

**Chess**

- Kasparov vs.
  - World, Deep Blue II
- Endgames, tablebases
  - Stiller, Nalimov
  - Combinatorial GT applied
    - Values found [Elkies, 1996]
  - SETI@Home parallel power to build database?
- Historical analysis...

White to move, wins in move 243 with Rd7xNe7
Exciting Game Theory Research
Solving games

- 4x4x4 Tic-Tac-Toe [Patashnik, 1980]
- Connect-4 [Allen, 1989; Allis, 1988]
- Go-Moku [Allis et al., 1993]
  ◇ One of oldest games – boards found c. 1400 BC
- Checkers almost solved [Schaeffer, 1996]
Summary

- Combinatorial game theory, learned games
- Computational game theory, GAMESMAN
- Reviewed research highlights