Due Thursday, February 21st

1. (25 pts.) Big-O notation

The purpose of this problem is to teach you Big-O notation in a careful way. First, study the following.

Formally: If \( f(n), g(n) \) are two non-negative functions of a single integer variable, the statement \( f(n) \in O(g(n)) \) means that

\[
\exists N_0 \in \mathbb{N} . \exists C \in \mathbb{N} . \forall x \in \mathbb{N} . (x \geq N_0 \implies 0 \leq f(x) \leq C \cdot g(x)).
\]

In other words, \( O(g(n)) \) is the set of functions \( \{f_i(n) : \exists N_0 \in \mathbb{N} . \exists C \in \mathbb{N} . \forall x \in \mathbb{N} . x \geq N_0 \implies f_i(x) \leq C \cdot g(x)\} \). This is the definition of Big-O notation.

Informally: \( f(n) \in O(g(n)) \) means, roughly, that \( f(n) \) grows “no faster than” \( g(n) \) (except possibly for a constant factor), as \( n \) gets large. For instance, \( n^2 \in O(n^2) \), \( n(n+1)/2 \in O(n^2) \), and \( 10000n^2 \in O(n^2) \), because these functions all grow at asymptotically the same rate (ignoring constant factors). Also, \( n^2 \in O(n^3) \), because \( n^2 \) grows more slowly than \( n^3 \) does, as \( n \) gets large.

Some basic facts: If \( f(n) \in O(g(n)) \) and \( f'(n) \in O(g'(n)) \), then \( f(n) + f'(n) \in O(g(n) + g'(n)) \). If \( f(n) \in O(g(n)) \) and \( f'(n) \in O(g'(n)) \), then \( f(n) \times f'(n) \in O(g(n) \times g'(n)) \).

Common notation: Instead of writing \( f(n) \in O(g(n)) \), almost everyone instead writes \( f(n) = O(g(n)) \). Strictly speaking, this is a sloppy abuse of notation, but this practice is widespread; you are guaranteed to see it throughout your studies of computer science, so be prepared. Also, we often write something like \( n^2 \) as a shorthand for the function \( f(n) = n^2 \), just to make our life easier.

Now, with that background established, do the following problems:

1. Prove that \( n^2 + 2008 \in O(n^3) \).
   
   \textit{Hint:} One possible approach is to give an example of constants \( N_0, C \) that satisfy the definition.

2. Prove that \( 77n^3 \log n \in O(n^4) \).

3. True or false: There exists \( e \in \mathbb{N} \) such that \( 2^n \in O(n^e) \). Briefly justify your answer.

4. Prove that if \( f(n) \in O(g(n)) \) and \( g(n) \in O(h(n)) \), then \( f(n) \in O(h(n)) \).

5. Critique the following argument. Is the reasoning valid? If not, why not? If there is an error, identify the erroneous step and explain what’s wrong with it.

   We have \( n^2 = O(n^4) \).
   Also, we have \( n^2 = O(n^3) \).
   By transitivity, it follows that \( O(n^4) = O(n^3) \).
   This means that \( n^4 = O(n^3) \).

2. (25 pts.) Another pairing problem

In this problem, we will study a matching problem that is somewhat different from the stable marriage problem. As before, we have \( n \) men and \( n \) women, but there are no preference lists. Instead, we have a list
of potential couples \((m, w)\) who are considered “compatible” (where \(m\) is a man and \(w\) a woman). The task is to find a pairing of all the men and women, subject to the restriction that a man and woman can only be paired together if they are compatible.

In other words, we are given a compatibility set \(C\), where \((m, w) \in C\) means that \((m, w)\) are a compatible couple. We say that a pairing is a compatible pairing if it is made solely out of compatible couples from \(C\). Given \(C\), the goal is to find a compatible pairing. (Any such pairing will do. Since we don’t have preference lists, we don’t have to worry about stability.)

If \(S\) is a set of some number of men, let \(f(S)\) denote the set of women who are compatible with some man in \(S\), i.e., \(f(S) = \{w : \exists m \in S. (m, w) \in C\}\). We’ll say that the compatibility set \(C\) is plentiful if for every set \(S\) of at most \(n\) men, \(|f(S)| \geq |S|\). (Recall that \(|S|\) denotes the size of the set \(S\), i.e., the number of elements it has.) We’ll say that the compatibility set \(C\) is super-plentiful if for every non-empty set \(S\) of at most \(n - 1\) men, \(|f(S)| > |S|\).

In this problem, you will prove that a compatible pairing exists if and only if \(C\) is plentiful. (There exist efficient algorithms to find such a pairing, if it exists, but they are too complex to cover here.)

1. Not every \(C\) has a compatible pairing. Prove that if \(C\) has a compatible pairing, then \(C\) is plentiful.

2. Suppose that \(C\) is plentiful. Let \((a, b)\) be a compatible couple from \(C\), i.e., \((a, b) \in C\). Suppose that \(a\) and \(b\) fall madly in love, elope, and fly off to Hawaii for their honeymoon. We are left with \(n - 1\) men, \(n - 1\) women, and the compatibility set \(C' = \{(m, w) \in C : m \neq a \land w \neq b\}\). Are we guaranteed that \(C'\) is necessarily plentiful? Briefly justify your answer.

3. Same as in part 2, except now we are told that \(C\) is super-plentiful. Prove that, in this case, \(C'\) is guaranteed to be plentiful.

4. Prove that if \(C\) is plentiful but not super-plentiful, then there exists a non-empty set \(M\) of men such that \(|f(M)| = |M|\) and \(|M| < n\).

5. Suppose that \(C\) is plentiful, and moreover that there is a non-empty set \(M\) of men such that \(|f(M)| = |M|\) and \(|M| < n\). Let \(k = |M|\). Suppose that those \(k\) men \((M)\) and those \(k\) women \((f(M))\) hop aboard a space ship to Mars, leaving behind \(n - k\) men and \(n - k\) women here on Earth. Let \(C_M = \{(m, w) \in C : m \in M \land w \in f(M)\}\) and \(C_E = \{(m, w) \in C : m \in M \land w \notin f(M)\}\). Prove that \(C_M\) is plentiful.

6. Same as in part 5, but now prove that \(C_E\) is plentiful.

    Hint: If \(S\) is a set of male earthlings, what can you say about \(|f(S \cup M)|\) vs \(|f(S) \setminus f(M)|\)?

7. Prove that if \(C\) is plentiful, then it has a compatible pairing.

    Hint: Use strong induction and the previous parts.