CS 70 Fall 2003 — Discrete Mathematics for CS — Wagner — HW 11

Due Thursday December 4

1. **(25 pts.) Independence**

   (a) Show that, for independent random variables X, Y, we have $\mathbb{E}[XY] = \mathbb{E}[X] \times \mathbb{E}[Y]$.
   HINT: Show first—carefully!—that, even if the r.v.'s are *not* independent, $\mathbb{E}[XY] = \sum_a \sum_b ab \times \Pr[X = a \wedge Y = b]$.

   (b) Give a simple example to show that the conclusion of the previous part is not necessarily true when X and Y are not independent.

2. **(25 pts.) Countability** ***New! (11/23)***
   Say that a function $f : \mathbf{N} \to \{0,1\}$ has *finite support* if it is non-zero on only a finite set of inputs. Let S denote the set of functions $f : \mathbf{N} \to \{0,1\}$ with finite support.

   Prove that S is countably infinite.

   HINT: Find a bijection between S and an appropriate subset of $\mathbf{N}$.

3. **(25 pts.) Fairy dust: The ghost of Cantor past**
   The Cantor set is an amazing object. It is defined iteratively as follows. Start with the interval $[0,1]$ of real numbers between zero and one. In the first iteration, delete the middle third of this interval, namely, $(1/3, 2/3)$. You're left with the union of two smaller line segments, namely, $[0,1/3] \cup [2/3,1]$. In the second iteration, delete the middle third of these two segments. Continue forever. In the n-th iteration, you remove the middle third of each of the 2^{n-1} segments left over from the previous iteration. Let S denote the set of points that are left over after iterating forever, i.e., $x \in S$ if x is not removed in any iteration.

   (a) Argue that the total length of the intervals left over after n iterations is $(2/3)^n$.

   (b) Argue that, if we choose a real number X uniformly at random from the interval $[0,1]$, then $\Pr[X \in S] = 0$.

   Regarding part (b) of the above question, you might be amused by the following quotation:

   > "When I was a freshman, a graduate student showed me the Cantor set, and remarked that although there were supposed to be points in the set other than the endpoints, he had never been able to find any. I regret to say that it was several years before I found any for myself." —Ralph P. Boas, Jr, *Lion Hunting & Other Mathematical Pursuits*, 1995.

4. **(25 pts.) Cantor's revenge**
   In the previous question, you showed that the Cantor set is, in some sense, very small. In this question, you're going to show that the Cantor set is, in another sense, very large.

   Show that the Cantor set is uncountably infinite.

HINT: There are two standard ways to prove that something is uncountable: Find a bijection between it and some other uncountable set; or, use diagonalization.

Also, you might find it useful to know the following alternative definition of the Cantor set: S is the set of real numbers $x \in [0,1]$ that can be represented in base 3 (trinary) using only 0's and 2's (i.e., no 1's). (Be warned that there is some ambiguity in trinary representations: $1/3$ could be represented as either $0.10000\ldots$ or $0.02222\ldots$. For this definition, we require that the ambiguity be resolved by always using representations that end in $02222\ldots$ rather than $10000\ldots$, whenever you have a choice.)