Schottky Algorithms: Classical meets Tropical

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From curves to Jacobians

Let $C$ be a curve over $\mathbb{C}$ of genus $g$. The Jacobian of $C$ is a complex torus $\mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$, where $\tau$ is the Riemann matrix.

1. Choose a homology basis of $2g$ cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$.
2. Choose a basis of $g$ holomorphic differentials $\omega_1, \ldots, \omega_g$.
3. Compute the $g \times 2g$ matrix $(A | B)$, where $A_{ij} = \oint_{a_i} \omega_j$ and $B_{ij} = \oint_{b_i} \omega_j$.
4. The Riemann matrix is $A^{-1}B$.  

![Diagram of a hyperelliptic curve with a homology basis showing $a_1, a_2, b_1, b_2$.](image)
Example: Elliptic curves

Elliptic curves have genus 1. The points of an elliptic curve form an abelian group, which is isomorphic to its Jacobian.
Schottky problem

Every Riemann matrix is in the Siegel upper-half space

\[ \mathcal{H}_g = \{ \tau \text{ complex symmetric } g \times g \text{ matrix }, \text{Im} \tau > 0 \}. \]

The symplectic group \( \text{Sp}(2g, \mathbb{Z}) \) acts on \( \mathcal{H}_g \) by

\[ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \gamma \cdot \tau = (A \tau + B)(C \tau + D)^{-1} \]

The moduli space of principally polarized abelian varieties of dimension \( g \) is

\[ \mathcal{A}_g = \mathcal{H}_g / \text{Sp}(2g, \mathbb{Z}). \]

The Torelli map sends a curve of genus \( g \) to its Jacobian:

\[ J: \mathcal{M}_g \longrightarrow \mathcal{A}_g, \quad [C] \mapsto [\mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)]. \]

The image \( \mathcal{J}_g \) is the Schottky locus.
Schottky problem

**Schottky problem**: Which abelian varieties of dimension $g$ are Jacobians of genus $g$ curves?

\[
\dim(J_g) = \dim(M_g) = 3g - 3, \\
\dim(A_g) = \dim(H_g) = \binom{g + 1}{2}.
\]

- For $g \leq 3$, every abelian variety is a Jacobian.
- For $g = 4$, the dimensions are 9 and 10 so $J_4 \not\subset H_4$.
  - Solution by Igusa using theta functions.
- For $g \geq 5$, Schottky problem is open.
Theta functions and theta constants

The Riemann theta function is the holomorphic function

\[ \theta : \mathbb{C}^g \times \mathcal{H}_g \rightarrow \mathbb{C}, \quad \theta(z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp \left( \pi i n^T \tau n + 2\pi i n^T z \right) \]

A characteristic is \( m \in \mathbb{F}_2^{2g} \), which we write as \( m = \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}, \epsilon, \delta \in \mathbb{F}_2^g \).

The Riemann theta function with characteristic \( m \) is

\[ \theta[m](\tau, z) = \sum_{n \in \mathbb{Z}^g} \exp \left[ \pi i \left( n + \frac{\epsilon}{2} \right)^T \tau \left( n + \frac{\epsilon}{2} \right) + 2\pi i \left( n + \frac{\epsilon}{2} \right)^T \left( z + \frac{\delta}{2} \right) \right] \]

The theta constants are

\[ \theta[m](\tau) := \theta[m](0, \tau) \]
Schottky problem in genus 4

For azygetic characteristics $m_1, m_2, m_3 \in \mathbb{F}_2^8$ and a size 8 subgroup $N \subset \mathbb{F}_2^8$ such that all $m_i + N$ are even, define

$$\pi_i = \prod_{m \in m_i + N} \theta[m](\tau, 0) \quad \text{for} \quad i = 1, 2, 3.$$ 

**Theorem (Igusa)**

*The function $\mathcal{H}_4 \to \mathbb{C}$ that takes a $4 \times 4$-matrix $\tau$ to the Schottky–Igusa modular form*

$$\pi_1^2 + \pi_2^2 + \pi_3^2 - 2\pi_1\pi_2 - 2\pi_1\pi_3 - 2\pi_2\pi_3$$

vanishes if and only if $\tau$ lies in the Schottky locus.
Example: Schottky Decision

We use the Sage program abelfunctions for numerical computations. The plane curve $y^5 + x^3 - 1 = 0$ has genus four, with Riemann matrix $\tau$:

$$
\begin{pmatrix}
0.169 + 1.417i & -0.817 - 0.251i & -0.056 - 0.448i & 0.247 + 0.363i \\
-0.817 - 0.251i & -0.313 + 0.671i & -0.028 - 0.572i & 0.341 + 0.403i \\
-0.056 - 0.448i & -0.028 - 0.572i & 0.324 + 1.450i & -0.965 - 0.638i \\
0.247 + 0.363i & 0.341 + 0.403i & -0.965 - 0.638i & 0.624 + 0.737i
\end{pmatrix}.
$$

Evaluating the 16 theta constants $\theta[m](\tau, 0)$ numerically, we find

$$
\pi_1^2 + \pi_2^2 + \pi_3^2 = -5.1347288827 + 6.1388787058i, \\
2(\pi_1 \pi_2 + \pi_1 \pi_3 + \pi_2 \pi_3) = -5.1347288264 + 6.1388793144i.
$$

We conclude that $\tau$ lies in the Schottky locus as expected.
Schottky Recovery problem

Let \( \Theta(z) := \theta[0](\tau, z) \). We compute a singular point \( z^* \in \mathbb{C}^4 \) of the theta divisor \( \Theta^{-1}(0) \) by solving:

\[
\Theta(z) = \frac{\partial \Theta}{\partial z_1}(z) = \frac{\partial \Theta}{\partial z_2}(z) = \frac{\partial \Theta}{\partial z_3}(z) = \frac{\partial \Theta}{\partial z_4}(z) = 0.
\]

The Taylor series of \( \Theta \) at \( z^* \) is

\[
\Theta(z^* + x) = f_2(x) + f_3(x) + f_4(x) + \text{higher order terms},
\]

where the \( f_s \) are homogeneous polynomials of degree \( s \) in \( x \).

**Proposition (Kempf)**

*The canonical curve with Riemann matrix \( \tau \) is the degree 6 curve in \( \mathbb{P}^3 \) that is defined by \( f_2 = 0 \) and \( f_3 = 0 \).*
Example: Schottky Recovery

Let $\tau \in J_4$ be the Riemann matrix of $C = \{ x^3y^3 + x^3 + y^3 = 1 \}$. Using abelfunctions and SciPy, we compute $\tau$ and $z^* = (0.555 + 0.698i, 0.537 + 0.269i, -0.500 - 0.590i, 0.555 + 0.698i)$.

\[
\begin{align*}
f_2(x) &= (-3.045 + 21.981i) \cdot x_1^2 + (-237.952 + 252.547i) \cdot x_1 x_2 \\
&\quad + (-222.356 + 139.956i) \cdot x_1 x_3 + (-200.669 - 16.596i) \cdot x_1 x_4 \\
&\quad + (-191.162 - 85.227i) \cdot x_2^2 + (-429.114 + 167.321i) \cdot x_2 x_3 \\
&\quad + (-237.952 + 252.547i) \cdot x_2 x_4 + (-206.759 + 27.365i) \cdot x_3^2 \\
&\quad + (222.356 + 139.956i) \cdot x_3 x_4 + (-3.045 + 21.981i) \cdot x_4^2 \\
f_3(x) &= (441.376 + 61.141i) \cdot x_1^3 + (2785.727 + 2303.609i) \cdot x_1^2 x_2 \\
&\quad + \cdots \cdots + (441.376 + 61.141i) \cdot x_4^3.
\end{align*}
\]
A weighted metric graph is $\Gamma = (V, E, l, w)$, with vertex set $V$, edge set $E$, a length function $l : E \to \mathbb{R}_{>0}$, and a weight function $w : V \to \mathbb{Z}_{\geq 0}$. The genus of $\Gamma$ is

$$g = |E| - |V| + 1 + \sum_{v \in V} w(v).$$

Examples of genus 3 weighted metric graphs:

2\,○\,\, 1\,→\,1\, 2\, 2\,→\,1\, 3

The moduli space $\mathcal{M}_g^{\text{trop}}$ comprises all metric graphs of genus $g$. The tropical Torelli map $\mathcal{M}_g^{\text{trop}} \to \mathcal{H}_g^{\text{trop}}$ takes $\Gamma$ to its Riemann matrix $Q_\Gamma$, which is symmetric and positive semidefinite.
Choose the spanning tree $T = \{e_2, e_3, e_4\}$, with cycle basis $\omega_1 = e_1 + e_3 + e_2$, $\omega_2 = -e_3 + e_5 + e_4$, $\omega_3 = -e_2 - e_4 + e_6$. Let

$$B = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \quad D = \text{Diag}(13, 7, 11, 2, 5, 3)$$

The Riemann matrix is then

$$Q_T = BDB^T = \begin{pmatrix} 22 & -7 & -13 \\ -7 & 23 & -11 \\ -13 & -11 & 27 \end{pmatrix}.$$
The tropical Schottky locus $\mathcal{J}_g^{\text{trop}}$ is the set of all matrices

$$Q_{\Gamma} = BDB^T = l(e_1)b_1b_1^T + l(e_2)b_2b_2^T + \cdots + l(e_m)b_mb_m^T.$$ 

The cone of all Riemann matrices for a graph $G$ is

$$\sigma_{G,B} = \mathbb{R}_{>0}\{b_1b_1^T, b_2b_2^T, \ldots, b_mb_m^T\}.$$ 

The cones $\sigma_{G,B}$ form a polyhedral fan whose support is the tropical Schottky locus $\mathcal{J}_g^{\text{trop}}$.

- Given $Q \in \mathcal{H}_g^{\text{trop}}$, how can we determine if $Q \in \mathcal{J}_g^{\text{trop}}$?
- Given $Q \in \mathcal{J}_g^{\text{trop}}$, how can we compute $\sigma_{G,B}$ such that $Q \in \sigma_{G,B}$?
Delaunay subdivisions

Let $Q \in \mathcal{H}_g^{\text{trop}}$, consider the quadratic form

$$l_Q : \mathbb{Z}^g \longrightarrow \mathbb{Z}^g \times \mathbb{R}, \quad x \mapsto (x, x^T Q x).$$

Take the convex hull of the image of $l_Q$ in $\mathbb{R}^g \times \mathbb{R} \cong \mathbb{R}^{g+1}$. By projecting down the lower faces, we obtain a periodic dicing of the lattice $\mathbb{Z}^g \subset \mathbb{R}^g$, called the Delaunay subdivision $\text{Del}(Q)$.

Example: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $l_Q : (x, y) \mapsto (x, y, x^2)$. 
Voronoi decomposition

Dual to the Delaunay subdivision is the Voronoi decomposition of \( \mathbb{R}^g \), which consists of lattice translates of the Voronoi polytope.
Voronoi Decomposition

Fix an arbitrary Delaunay subdivision $D$. Its secondary cone is

$$\sigma_D = \{ Q \in \mathcal{H}_g^{\text{trop}} \mid \text{Del}(Q) = D \}.$$  

**Theorem (Voronoi)**

*The cones $\sigma_D$ form a polyhedral fan, called the second Voronoi decomposition of $\mathcal{H}_g^{\text{trop}}$. There are only finitely many secondary cones $\sigma_D$ up to the action of $\text{GL}_g(\mathbb{Z})$.***

The tropical Schottky locus is a subfan of the second Voronoi decomposition. Given a graph $G$ with metric $D$, homology basis $B$, and Riemann matrix $Q = BDB^T$,

$$\sigma_{\text{Del}(Q)} = \sigma_{G,B} = \mathbb{R}_{>0}\{b_1b_1^T, b_2b_2^T, \ldots, b_mb_m^T\}.$$
Tropical Schottky Decision for \( g = 4 \)

In his thesis, Frank Vallentin lists all 52 combinatorial types of Delaunay subdivisions of \( \mathbb{Z}^4 \).

**Lemma (CKS)**

The f-vectors of the 16 Voronoi polytopes representing the Schottky locus \( \mathcal{J}^\text{trop}_4 \) are distinct from the f-vectors of the other 36 Voronoi polytopes, corresponding to \( \mathcal{H}^\text{trop}_4 \setminus \mathcal{J}^\text{trop}_4 \).

This gives an algorithm to decide if a matrix is in the tropical Schottky locus.
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<tr>
<th>Graph $G$</th>
<th>$f_0$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>Dimension of $\sigma_D$</th>
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Example: Tropical Schottky Recovery

\[ Q = \begin{pmatrix} 17 & 5 & 3 & 5 \\ 5 & 19 & 7 & 11 \\ 3 & 7 & 23 & 16 \\ 5 & 11 & 16 & 29 \end{pmatrix}. \]

Using polyhedral, we compute the f-vector \((96, 198, 130, 28)\). Hence \(Q \in J_4^{\text{trop}}\), and \(G\) is the triangular prism. We find \(X \in \text{GL}_4(\mathbb{Z})\) that maps \(Q\) into our preprocessed secondary cone:

\[ X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} \in \text{GL}_4(\mathbb{Z}) \quad \text{gives} \]

\[ Q' = X^T Q X = \begin{pmatrix} 26 & 9 & -9 & 0 \\ 9 & 20 & 7 & -2 \\ -9 & 7 & 23 & 3 \\ 0 & -2 & 3 & 17 \end{pmatrix} \in \sigma_{G,B}. \]
Example: Tropical Schottky Recovery (continued)

$Q'$ is the Riemann matrix of this metric graph, with basis cycles $e_2 + e_6 - e_3$, $-e_1 - e_4 + e_7 + e_2$, $-e_1 - e_5 + e_8 + e_3$, and $e_4 + e_9 - e_5$. These are the rows of the $4 \times 9$-matrix $B$. We compute $D = \text{diag}(\ell_1, \ldots, \ell_9) = \text{diag}(7, 9, 9, 2, 3, 8, 2, 4, 12)$. 

![Graph illustration]
Schottky locus of a Spectrahedron

A spectrahedron is the intersection of $H_{trop}^g$ with an affine-linear space $L$ of symmetric matrices. The Schottky locus of a spectrahedron is $J_{trop}^g \cap L$.

$$Q = \begin{bmatrix}
1589 - 2922s + 960t & 789 - 1322s & -802 + 660s - 1350t & -820 + 3260s + 2550t \\
789 - 1322s & 1589 - 2922s - 960t & -820 + 3260s - 2550t & -820 + 660s + 1350t \\
-202 + 660s - 1350t & -820 + 3260s - 2550t & 1665 + 450s + 3120t & -25 - 2930s \\
-820 + 3260s + 2550t & -820 + 660s + 1350t & -25 - 2930s & 1665 + 450s - 3120t
\end{bmatrix}.$$
Let $Q \in \mathcal{H}_g^{\text{trop}}$ be a positive definite matrix for arbitrary $g$. Mikhalkin and Zharkov define the tropical theta function:

$$\Theta(Q, x) := \max_{\lambda \in \mathbb{Z}^g} \{\lambda^T Q x - \frac{1}{2} \lambda^T Q \lambda\}.$$ 

This describes the asymptotic behavior of the classical Riemann theta function with Riemann matrix $t \cdot \tau$ when $t$ goes to infinity, where $Q$ is the imaginary part of $\tau$.

We define the tropical theta constant with characteristic $u \in \mathbb{F}_2^g$

$$\Theta_u(Q) = \max_{\lambda \in \mathbb{Z}^g} \{-\left(\lambda + \frac{u}{2}\right)^T Q \left(\lambda + \frac{u}{2}\right)\}.$$
Theta matroid

For $v \in \mathbb{F}_2^g$ consider

$$\vartheta_v(Q) := \sum_{u \in \mathbb{F}_2^g} (-1)^{u^T v} \cdot \Theta_u(Q).$$

The theta matroid $M(Q)$ is the binary matroid represented by

$$\{v \in \mathbb{F}_2^g : \vartheta_v(Q) \neq 0\}.$$

Theorem (CKS)

If $Q \in \mathcal{J}_g^{\text{trop}}$ then $M(Q)$ is cographic. In that graph, assign the length $2^{3-g} \cdot \vartheta_v(Q)$ to the edge labeled $v$. The resulting metric graph has Riemann matrix $Q$.

This gives another algorithm for Tropical Schottky Recovery.
Tropical meets Classical

Recall the classical Schottky–Igusa modular form:

\[ \pi_1^2 + \pi_2^2 + \pi_3^2 - 2(\pi_1\pi_2 + \pi_1\pi_3 + \pi_2\pi_3) \]

where each \( \pi_i \) is a product of 8 theta constants.

Tropical Schottky–Igusa modular form

\[
\max_{i,j=1,2,3} (\pi_i^{\text{trop}} + \pi_j^{\text{trop}}),
\]

where \( \pi_i^{\text{trop}} = \sum_{m \in m_i + N} \Theta_{m'}(Q), \ Q = \text{im}(\tau). \)

This is a piecewise-linear convex function \( \mathcal{H}_4^{\text{trop}} \to \mathbb{R} \). Its breakpoint locus is the set all \( Q \) where the maximum is attained twice.
Tropical Igusa locus

We say that $m_1, m_2, m_3, N$ is admissible if $N \subset \mathbb{F}_2^8$ has rank three, the triple $\{m_1, m_2, m_3\} \subset \mathbb{F}_2^8$ is azygetic, all elements of $m_i + N$ are even, and $N' \subset \mathbb{F}_2^4$ also has rank three.

The tropical Igusa locus is the intersection, over all admissible choices $m_1, m_2, m_3, N$, of the breakpoint loci.

**Theorem (CKS)**

A matrix $Q \in \mathcal{H}_4^{\text{trop}}$ lies in the tropical Igusa locus if and only if $\vartheta_v(Q) \geq 0$ for all $v \in \mathbb{F}_2^4$. This locus strictly contains the tropical Schottky locus $\mathcal{J}_4^{\text{trop}}$. 
Open problems

1. Can the tropical Schottky locus $\mathcal{J}_4^{\text{trop}}$ be cut out by additional tropical modular forms?

2. Can we generalize our results to genus 5 and above?

3. Can we make the inverse tropical Torelli map continuous?
The code for the computations are available on my website:

https://www.cs.berkeley.edu/~chualynn/schottky/