DOUBLE SCROLL AND CELLULAR NEURAL NETWORKS

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Abstract – This paper reports a chaotic attractor with an autonomous three-cell Cellular Neural Network (CNN). It is shown that the attractor has a structure very similar to the *double scroll* attractor. Through some equivalent transformations this circuit in three major subspaces of its state space is shown to belong to *Chua's circuit family*, although originating from a completely different field.

1 Introduction

It has been long recognized that both real and mathematical models of neural systems can give rise to complex nonperiodic ("chaotic") dynamics. It is also recognized that brains are nonlinear networks composed of chaotic subsystems [4]. Therefore, it is important to investigate dynamical behaviors of minimalsized networks, which exhibit chaos. Unfortunately such chaotic attractors observed in continuous analog models of neural networks are rarely reported.

Recently, chaos has been observed in a two-cell nonreciprocal CNN with sinusoidal excitation [3]. For autonomous analog circuits three cells are needed to generate complex dynamic behavior. In this paper such a chaotic attractor is shown with a three-cell CNN. This attractor has a surprising similarity to the *double scroll* attractor [2]. In fact, with some approximations and equivalent transformations this three-cell CNN can be interpreted as a circuit, which in its major operating subspaces belongs to Chua's circuit family.

First the system equations are given and some related aspects are discussed in Section 2 briefly. Some simulated results from the proposed system are also given in this section. In Section 3 relations of this circuit to Chua's circuit family will be discussed.

2 System Description

Consider the CNN of Fig. 1(a), where the function $f(\cdot)$ is graphically shown by Fig.1(b). The dynamics of the system can be described by the set of ordinary differential equations:

$$\begin{aligned} \dot{x}_1 + x_1 &= p_1 f(x_1) - s f(x_2) - s f(x_3) \\ \dot{x}_2 + x_2 &= -s f(x_1) + p_2 f(x_2) - r f(x_3) \\ \dot{x}_3 + x_3 &= -s f(x_1) + r f(x_2) - p_3 f(x_3) \end{aligned} \quad (1)$$

with the output function

$$f(x_i(t)) = \frac{1}{2}(|x_i(t) + 1| - |x_i(t) - 1|) \qquad (i = 1, 2, 3) \quad (2)$$

where $p_1 > 1$, $p_2 > 1$, $p_3 \geq 1$, $r > 0$, $s > 0$.

One may notice that a normalized description is used for convenience, and the inputs v_{uij}'s and bias currents I_{ij}'s are all omitted.

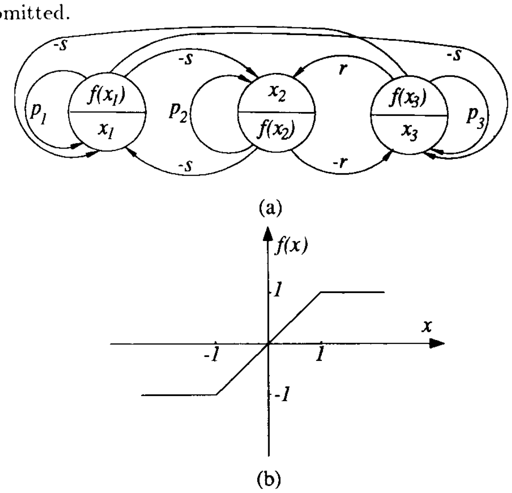

Figure 1. (a) The autonomous three-cell CNN ;
(b) The output function

Fig. 2 shows the chaotic attractor observed by solving (1) with the following parameter set

$$p_1 = 1.25, \quad p_2 = 1.1, \quad p_3 = 1, \quad s = 3.2, \quad r = 4.4 \quad (3)$$

and initial condition

$$\boldsymbol{x}(0) = [\ 0.1\ \ 0.1\ \ 0.1\ ]^T \ .$$

It is obvious that Fig. 2 shows an amazing similarity to the double scroll attractor. System (1), (2) along with the parameter set (3) has only three equilibria, which are all unstable. Therefore, the unstable solution of the system is not surprising. It will be shown that the eigenspaces of the equilibria have the same structure as the double scroll, so that the strong similarity is not a accidental phenomenon.

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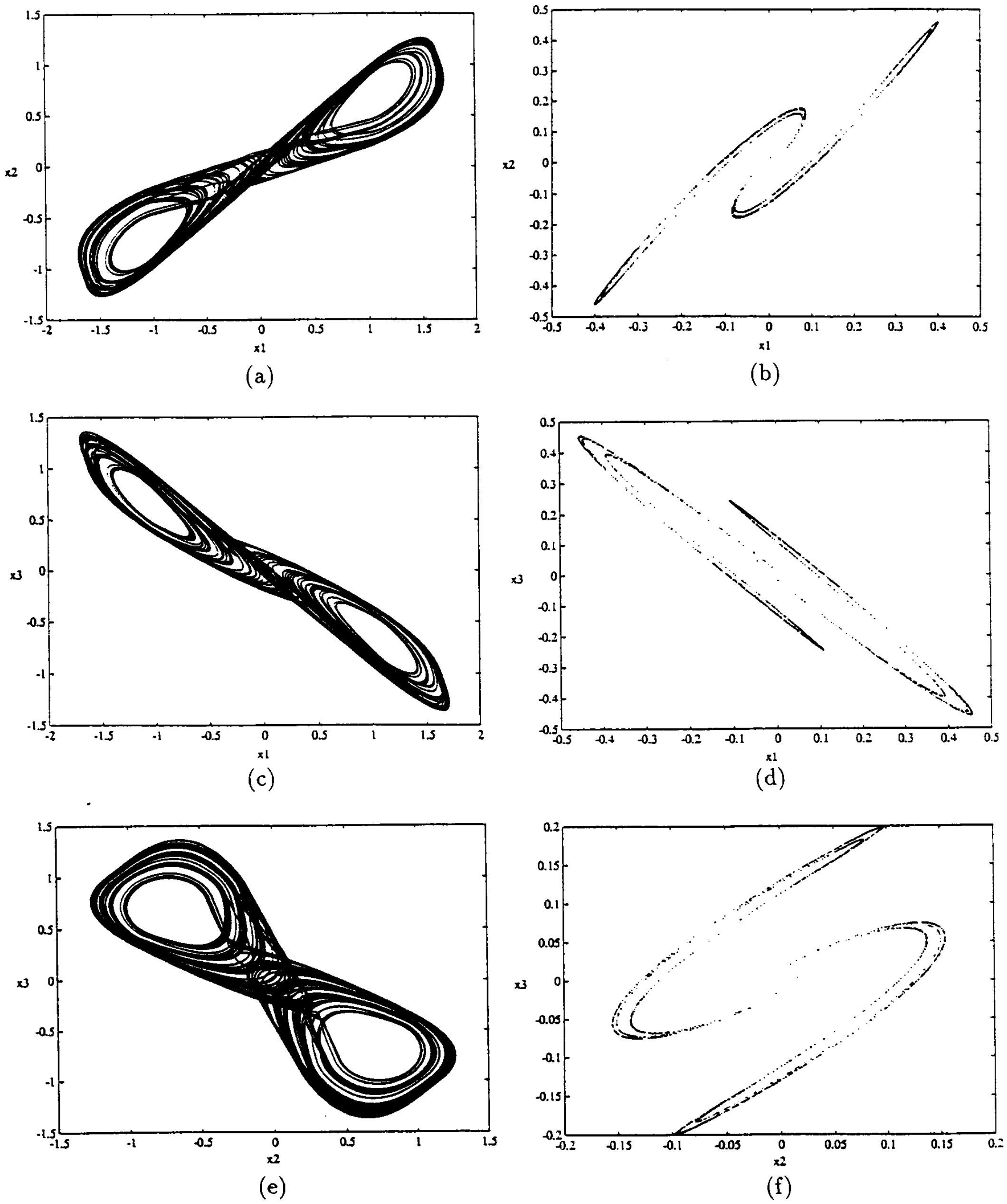

Fig. 2 The chaotic attractor. (a) Projection onto the (x_1, x_2)-plane. (b) Cross section of the (x_1, x_2)-plane with $x_3 = 0$. (c) Projection onto the (x_1, x_3)-plane. (d) Cross section of the (x_1, x_3)-plane with $x_2 = 0$. (e) Projection onto the (x_2, x_3)-plane. (f) Cross section of the (x_2, x_3)-plane with $x_1 = 0$.

Let us define the following three subsets of $\mathbb{R}^3$:

$$\begin{aligned} D_{+1} &= \{(x_1, x_2, x_3) \mid x_1 \geq 1,\ |x_2| \cdot |x_3| < 1\} \\ D_0 &= \{(x_1, x_2, x_3) \mid\ |x_1| \cdot |x_2| \cdot |x_3| < 1\} \\ D_{-1} &= \{(x_1, x_2, x_3) \mid x_1 \leq 1,\ |x_2| \cdot |x_3| < 1\}\ . \end{aligned} \quad (4)$$

With the parameters in (3), the equilibria are given by:

$$\begin{aligned} \boldsymbol{P}_{+1} &= (1.1971, 0.7273, -0.7107) \in D_{+1} \\ \boldsymbol{P}_0 &= (0, 0, 0) \in D_0 \\ \boldsymbol{P}_{-1} &= (-1.1971, -0.7273, 0.7107) \in D_{-1} \end{aligned} \quad (5)$$

The eigenvalues of Jacobi-matrix at these equilibria, and hence in the whole subspaces respectively, are calculated as:

$$\begin{aligned} \gamma_p &= -1.0;\ \sigma_p \pm j\omega_p = 0.05 \pm j4.3997 \text{ in } D_{+1} \text{ and } D_{-1} \\ \gamma_0 &= 1.935\ \ \sigma_0 \pm j\omega_0 = -0.7925 \pm j1.1593 \text{ in } D_0 \end{aligned}$$

Let $E^s(\boldsymbol{P}_\pm)$ be the eigenspace corresponding to the real eigenvalue γ_p at $\boldsymbol{P}_\pm$ and let $E^u(\boldsymbol{P}_\pm)$ be the eigenspace corresponding to the complex eigenvalues $\sigma_p \pm j\omega_p$ at $\boldsymbol{P}_\pm$. Following the same way, we define $E^u(\boldsymbol{P}_0)$ and $E^s(\boldsymbol{P}_0)$ corresponding to γ_0 and $\sigma_0 \pm j\omega_0$, respectively. Then,

$$\begin{aligned} \dim E^s(\boldsymbol{P}_\pm) &= \dim E^u(\boldsymbol{P}_0) = 1 \\ \dim E^u(\boldsymbol{P}_\pm) &= \dim E^s(\boldsymbol{P}_0) = 2 \end{aligned}$$

Now we see that the eigenspaces of the equilibria have almost identical structures with the double scroll, this gives some reasons, why both attractors share nearly the same appearance and form.

3 Relations to Chua's Circuit

Let us imagine that the output variables of cell 2 and 3 are not saturated, i.e. $f(x_2) = x_2$ and $f(x_3) = x_3$, (notice that this is the case in the subspaces D_{+1}, D_0, and D_{-1}). Then, because of the opposite-sign weights $+r$ and $-r$, which can be interpreted as a ideal gyrator, these two cells construct a oscillatory circuit. This transformation is shown in Fig. 3.

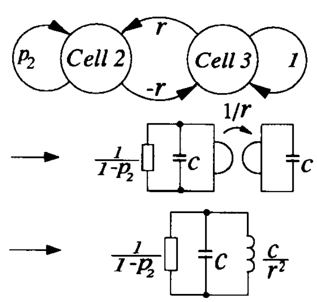

Figure 3. Circuit equivalent transformations of cell 2 and cell 3

On the other hand, cell 1 can be approximately substituted by a circuit with a linear capacitor and a nonlinear resistor (Fig. 4(a)). The nonlinear function of the resistance is given in Fig. 4(b). Now it is very obvious that the whole circuit shown in Fig. 5 really belongs to Chua's circuit family.

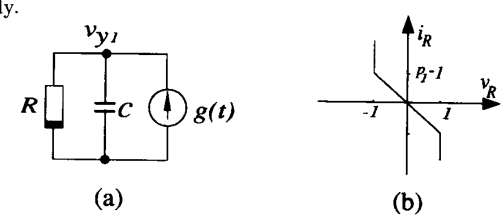

Figure 4. Approximation of the cell 1

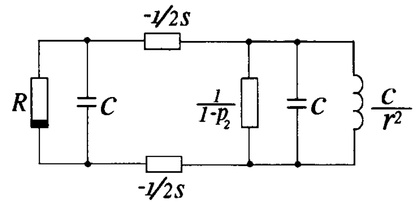

Figure 5. Approximation of the whole circuit

As one can see from Fig. 2, the output variables of cell 2 and 3 do not exceed the linear region ($|x_i| < 1$) for most operating time, so the approximation above gives a reasonable circuit-theoretic interpretation of the chaotic behavior. But it would be misleading to identify this attractor with the double scroll, because x_2 and x_3 do reach the saturation region ($|x_i| \geq 1$) for a not negligible time. It seems, that it is this saturation function that makes the chaotic behavior possible here. If $p_3 = 1$, $s = 3.2$ and $r = 4.4$ are fixed, the following relationship can be obtained: for a large p_2 (and consequently large p_1) one has a bad approximation and vice versa. An example to show this is given in Fig. 6.

4 Conclusion

An autonomous chaotic attractor with a three-cell CNN has been found, which has very similar structures with the double scroll.

References

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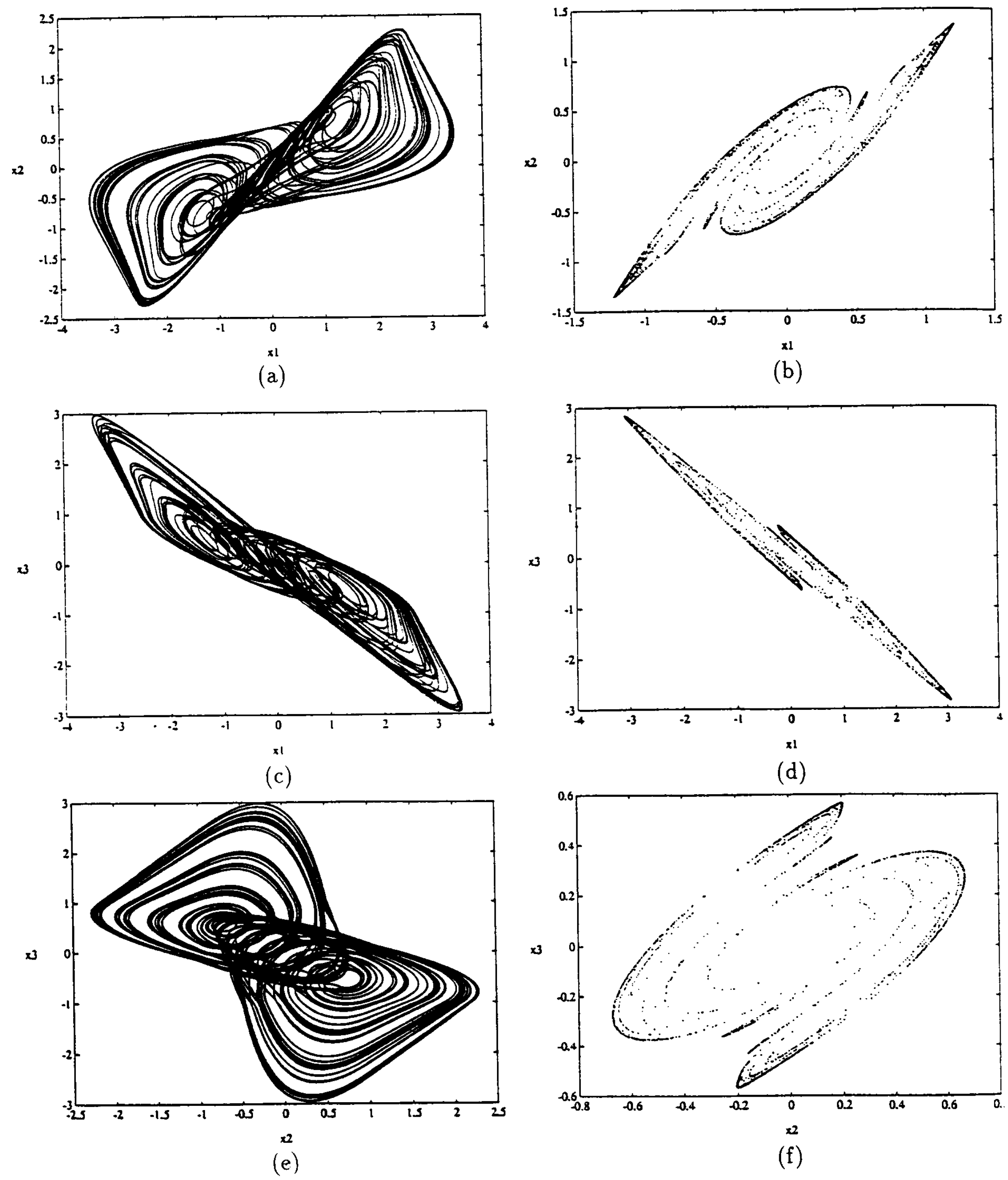

Fig. 6 The chaotic attractor with the parameters $p_1 = 1.99$, $p_2 = 2$, $p_3 = 1$, $s = 3.2$, $r = 4.4$; and initial states $\boldsymbol{x}(0) = [\ 0.1\ 0.1\ 0.1\ ]^T$. (a) Projection onto the (x_1, x_2)-plane. (b) Cross section of the (x_1, x_2)-plane with $x_3 = 0$. (c) Projection onto the (x_1, x_3)-plane. (d) Cross section of the (x_1, x_3)-plane with $x_2 = 0$. (e) Projection onto the (x_2, x_3)-plane. (f) Cross section of the (x_2, x_3)-plane with $x_1 = 0$.