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A Detailed Study of Adaptive Control of Chaotic Systems with Unknown Parameters

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Abstract. In this paper, we study the control of chaotic systems with unknown parameters. A stable adaptive control scheme is used to guarantee that the parameter estimator converges to stabilizing values such that the controlled chaotic system asymptotically approaches a reference point. A Lyapunov function approach is used to prove a global result which guarantees the stability of both controlled chaotic system and the adaptive parameter estimator. The center manifold theorem is used to prove the stability of the adaptive parameter estimator.

To demonstrate the usefulness of this adaptive control of chaotic systems, computer simulation results are provided. We use Chua's circuit with cubic nonlinearity in our simulations. We provide the simulation results of control of Chua's circuit with 6 unknown parameters.

1. Introduction

So far, there exist two main offsprings of applications of chaotic systems: control and synchronization. The control problem is motivated by some demands which need to regulate the chaotic systems into some desirable motion from different areas [3]–[5], [7]. The control problem is generally considered as that of stabilizing a chaotic system to an equilibrium point or a periodic orbit. The synchronization problem can be viewed as a special kind of control problem in which the goal is to track the desired chaotic trajectory [6], [8].

In real life applications, the parameters of a chaotic system are not always accessible. And the parameters may be time-varying [2], [9]. When some parameters of a chaotic system are unknown, we need to apply adaptive control techniques [1], [2], [11], [12], [13], [14] to control the chaotic system. In [1], the authors proposed a pool of adaptive controllers for purpose of compensating channel gain or mismatch of one parameter in a synchronization scheme. On the other hand, the authors of [2] employed some adaptive model-reference controllers to synchronize two chaotic systems with more than one unknown parameters. In [11]. the author used adaptive method to compensate the modeling error from observed chaotic time series data and achieved the control. In [12], the authors used an adaptive model-reference controller to achieve control from observed time series. The autoregressive self-tuning feedback method was used in [14].

In this paper, we study the control of chaotic system with more than one unknown parameter to an equilibrium point. To stabilize the controlled system, an adaptive parameter estimator is employed. Based on this adaptive estimator, a controller is used to stabilize the chaotic system to desired points. A Lyapunov function method is used to prove the asymptotic stability of the controlled chaotic system from a global point of view. Furthermore, the center manifold theory [19] is employed to get a detailed insight in the dynamics of the parameter estimator.

To demonstrate the usefulness of our adaptive method, the simulation results of control of Chua's circuit with cubic nonlinearity and with 6 unknown parameters are presented.

The organization of this paper is as follows. In section II, some theorems concerning about adaptive control of continuous chaotic system with unknown parameters are given. In section III, some simulation results using Chua's circuit with cubic nonlinearity are presented. In section IV, the conclusions are contained.

2. Control of Chaotic System with Unknown Parameters

The chaotic system we studied in this paper has the following form

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{\Phi}(\mathbf{x})\mathbf{\Theta} \tag{1}$$

where $\mathbf{x} \in R^n$, $\mathbf{\Phi}(\mathbf{x}) \in C^1(R^n, R^{n\times l})$ and $\mathbf{\Theta} \in R^l$. $\mathbf{\Theta}$ is the vector of unknown constant parameters. A is an $n \times n$ constant matrix. The $\mathbf{\Phi}(\mathbf{x}) = (\mathbf{\Phi}_{ij}(\mathbf{x}))$ are the matrix of smooth nonlinear functions take arguments in R^n and $\mathbf{\Phi}_{ij}(\mathbf{0}) = \mathbf{0}$. Since $\mathbf{\Phi}_{ij}(\mathbf{0}) = \mathbf{0}$, this system has an equilibrium point at the origin $\mathbf{x} = \mathbf{0}$, and the control objective is to globally stabilized this equilibrium for any unknown value of $\mathbf{\Theta}$. The controlled chaotic system is given by

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{\Phi}(\mathbf{x})\mathbf{\Theta} + \mathbf{u} \tag{2}$$

$\mathbf{u} \in R^n$ is the control vector. Since there exists an unknown parameter vector $\mathbf{\Theta}$ in the plant, we need to design a parameter estimator, $\hat{\mathbf{\Theta}}$, which asymptotically approaches the actual parameter vector $\mathbf{\Theta}$. We have the following theorem.

THEOREM 1 *Assume that there exists a matrix B such that $(A + B)$ is negative definite and the control law is given by*

$$\mathbf{u} = B\mathbf{x} - \mathbf{\Phi}(\mathbf{x})\hat{\mathbf{\Theta}} \tag{3}$$

where $\hat{\mathbf{\Theta}}$ is the estimated parameter vector, which is given by the following parameter update law

$$\dot{\hat{\mathbf{\Theta}}} = \mathbf{\Phi}^T(\mathbf{x})\mathbf{x} \tag{4}$$

Then we can draw the following two conclusions

1. $\mathbf{x}(\mathbf{t})$ and the parameter estimating error $\tilde{\mathbf{\Theta}} = \mathbf{\Theta} - \hat{\mathbf{\Theta}}$ can be made arbitrary small for all time $t \geq 0$, if we choose the initial error $\mathbf{e}(\mathbf{0})$ and $\tilde{\mathbf{\Theta}}(0)$ to be sufficiently small.

2. $\mathbf{x}(\mathbf{t}) \longrightarrow \mathbf{0}$ as $t \longrightarrow \infty$. Furthermore, if $\mathbf{x}$ is bounded for all time $t \geq 0$, then we have $\dot{\hat{\mathbf{\Theta}}} \longrightarrow 0$ as $t \longrightarrow \infty$.

Proof: We construct a Lyapunov function $V(\mathbf{x})$ as

$$V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{x} + \frac{1}{2}\tilde{\mathbf{\Theta}}^T\tilde{\mathbf{\Theta}} \tag{5}$$

Let $\tilde{\mathbf{\Theta}} = \mathbf{\Theta} - \hat{\mathbf{\Theta}}$, since $\mathbf{\Theta}$ is a constant parameter vector, we have

$$\dot{\tilde{\mathbf{\Theta}}} = -\dot{\hat{\mathbf{\Theta}}} \tag{6}$$

Differentiating V along the trajectories of the solutions of Eqs. (2) and (4), we get

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \mathbf{x}^T\dot{\mathbf{x}} + \tilde{\mathbf{\Theta}}^T\dot{\tilde{\mathbf{\Theta}}} \\ &= \mathbf{x}^T(A\mathbf{x} + B\mathbf{x} + \mathbf{\Phi}(\mathbf{x})\tilde{\mathbf{\Theta}}) - \tilde{\mathbf{\Theta}}^T\mathbf{\Phi}^T(\mathbf{x})\mathbf{x} \\ &= \mathbf{x}^T(A+B)\mathbf{x} + \mathbf{x}^T\mathbf{\Phi}(\mathbf{x})\tilde{\mathbf{\Theta}} - \tilde{\mathbf{\Theta}}^T\mathbf{\Phi}^T(\mathbf{x})\mathbf{x} \\ &= \mathbf{x}^T(A+B)\mathbf{x} \le 0 \end{aligned} \tag{7}$$

The equality is satisfied only when $\mathbf{x} = \mathbf{0}$. ∎

Remarks. From Theorem 1 we can see that the solutions of the controlled system and the update law converge to the following manifold

$$\mathsf{M} = \{(\mathbf{x}, \hat{\mathbf{\Theta}}) \in R^{n+l} | \mathbf{x} = \mathbf{0}\} \tag{8}$$

Since M is in fact the $\hat{\Theta}$-subspace in R^{n+l}, the following theorem is used to state which point the system converges to in the $\hat{\Theta}$-subspace.

THEOREM 2 *Suppose* $|\mathbf{\Phi}(\mathbf{x})| \le L|\mathbf{x}|$, *where* $L > 0$, *then there exists a constant vector* $\hat{\Theta}_\infty \in R^l$ *such that*

$$\lim_{t \longrightarrow \infty} \hat{\Theta}(t) = \hat{\Theta}_\infty \tag{9}$$

Proof: Theorem 1 guarantees that $\hat{\Theta}$ and $\mathbf{x}$ are bounded and from Eq. (7) we have

$$\dot{V}(\mathbf{x}, t) = \mathbf{x}^T(A+B)\mathbf{x} \le \overline{\lambda}_{A+B}|\mathbf{x}|^2 \tag{10}$$

where $\overline{\lambda}_{A+B}$ denotes the biggest eigenvalue of $(A+B)$. Since $(A+B)$ is negative definite we have $\overline{\lambda}_{A+B} < 0$. By Bolzano-Weierstrass theorem, there exists a sequence $\{t_i\}$ with $t_i \longrightarrow \infty$ as $i \longrightarrow \infty$ such that the sequence $\tilde{\mathbf{\Theta}}(t_i)$ has a limit $\tilde{\mathbf{\Theta}}_\infty \in R^l$, i.e.,

$$\tilde{\mathbf{\Theta}}(t_i) \longrightarrow \tilde{\mathbf{\Theta}}_\infty \ \textit{as}\ i \longrightarrow \infty \tag{11}$$

Let

$$\bar{\mathbf{\Theta}} = \tilde{\mathbf{\Theta}} - \tilde{\mathbf{\Theta}}_\infty = \hat{\mathbf{\Theta}}_\infty - \hat{\mathbf{\Theta}} \tag{12}$$

and for any $t \in [t_i, t_{i+1}]$, consider

$$\bar{\Theta}(t) = \bar{\Theta}(t_i) + \int_{t_i}^{t} \dot{\bar{\Theta}}(\tau)\, d\tau \tag{13}$$

In view of $|\Phi(\mathbf{x})| \leq L|\mathbf{x}(\mathbf{t})|$, we have

$$|\bar{\Theta}(t)| \leq |\bar{\Theta}(t_i)| + L \int_{t_i}^{t} |\mathbf{x}(\tau)|^2\, d\tau \tag{14}$$

In view of $\dot{V}(t) \leq \bar{\lambda}_{A+B}|\mathbf{x}|^2$ and $\bar{\lambda}_{A+B} < 0$, we have

$$\begin{aligned} \int_{t_i}^{t} |\mathbf{x}(\tau)|^2\, d\tau &\leq \frac{1}{\bar{\lambda}_{A+B}} \int_{t_i}^{t} \dot{V}(\tau)\, d\tau \\ &= -\frac{1}{\bar{\lambda}_{A+B}}(V(t_i) - V(t)) \end{aligned} \tag{15}$$

In view of $\dot{V} \leq 0$, we have $V(t_{i+1}) \leq V(t)$, then we have

$$\begin{aligned} \int_{t_i}^{t} |\mathbf{x}(\tau)|^2\, d\tau &\leq -\frac{1}{\bar{\lambda}_{A+B}}(V(t_i) - V(t_{i+1})) \\ &= -\frac{1}{2\bar{\lambda}_{A+B}}[(|\mathbf{x}(t_i)|^2 - |\mathbf{x}(t_{i+1})|^2) + (|\tilde{\Theta}(t_i)|^2 - |\tilde{\Theta}(t_{i+1})|^2)] \\ &\leq -\frac{1}{2\bar{\lambda}_{A+B}}[|\mathbf{x}(t_i)|^2 + (\tilde{\Theta}(t_i) - \tilde{\Theta}(t_{i+1}))^T(\tilde{\Theta}(t_i) + \tilde{\Theta}(t_{i+1}))] \\ &\leq -\frac{1}{2\bar{\lambda}_{A+B}}[|\mathbf{x}(t_i)|^2 + (\bar{\Theta}(t_i) - \bar{\Theta}(t_{i+1}))^T(\bar{\Theta}(t_i) + \bar{\Theta}(t_{i+1}) + 2\tilde{\Theta}_\infty)] \\ &\leq -\frac{1}{2\bar{\lambda}_{A+B}}[|\mathbf{x}(t_i)|^2 + (|\bar{\Theta}(t_i)| + |\bar{\Theta}(t_{i+1})|)(|\bar{\Theta}(t_i)| + |\bar{\Theta}(t_{i+1})| + 2|\tilde{\Theta}_\infty|)] \end{aligned} \tag{16}$$

Then from Eqs. (14) and (16), we have

$$|\bar{\Theta}(t)| \leq |\bar{\Theta}(t_i)| - \frac{L}{2\bar{\lambda}_{A+B}}[|\mathbf{x}(t_i)|^2 + (|\bar{\Theta}(t_i)| + |\bar{\Theta}(t_{i+1})|)(|\bar{\Theta}(t_i)| + |\bar{\Theta}(t_{i+1})| + 2|\tilde{\Theta}_\infty|)] \tag{17}$$

In view of Theorem 1, we know that $\lim_{t \longrightarrow \infty} \mathbf{x}(t) = \mathbf{0}$. And in view of Eq. (11), we know that the right hand side of Eq. (17) tends to zero as $t \longrightarrow \infty$, i.e.,

$$\lim_{t \longrightarrow \infty} \bar{\Theta}(t) = \lim_{t \longrightarrow \infty} (\hat{\Theta}_\infty - \hat{\Theta}) = \mathbf{0} \tag{18}$$

which means that $\lim_{t \longrightarrow \infty} \hat{\Theta} = \hat{\Theta}_\infty$. ■

Since any of the solutions of the adaptively controlled chaotic system converges to an equilibrium point on the manifold M, we would like to study the stability of an equilibrium point $(\mathbf{x}, \hat{\Theta}) = (\mathbf{0}, \hat{\Theta}_e)$, where $\hat{\Theta}_e \neq \Theta$. We have the following theorem.

THEOREM 3 *Consider the adaptively controlled chaotic system, concerning with the equilibrium point* $(\mathbf{x}, \hat{\Theta}) = (\mathbf{0}, \hat{\Theta}_e)$ *we can draw two conclusions*

1) $(\mathbf{0}, \hat{\Theta}_e)$ *is globally stable if all the eigenvalues of* $(A + B + \frac{\partial \mathbf{\Phi}(\mathbf{0})\tilde{\Theta}_e}{\partial \mathbf{x}})$ *is in the open left hand plane.*

2) $(\mathbf{0}, \hat{\Theta}_e)$ *is unstable if at least one eigenvalue of* $(A + B + \frac{\partial \mathbf{\Phi}(\mathbf{0})\tilde{\Theta}_e}{\partial \mathbf{x}})$ *is in the open right hand plane.*

Proof: Since $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$, then there exists a smooth vector-valued function $G_1(\mathbf{x}, \hat{\Theta}_e)$ with $G_1(\mathbf{0}, \hat{\Theta}_e) = \mathbf{0}$ and $\frac{\partial G_1(\mathbf{0}, \hat{\Theta}_e)}{\partial \mathbf{x}} = \mathbf{0}$, such that

$$\mathbf{\Phi}(\mathbf{x})\tilde{\Theta}_e = \frac{\partial \mathbf{\Phi}(\mathbf{0})\tilde{\Theta}_e}{\partial \mathbf{x}}\mathbf{x} + G_1(\mathbf{x}, \hat{\Theta}_e) \tag{19}$$

let

$$\bar{\Theta} = \hat{\Theta}_e - \hat{\Theta} = \tilde{\Theta} - \tilde{\Theta}_e \tag{20}$$

Then we have

$$\begin{aligned} \dot{\mathbf{x}} &= (A + B)\mathbf{x} + \mathbf{\Phi}(\mathbf{x})\tilde{\Theta} \qquad (21) \\ &= (A + B)\mathbf{x} + \mathbf{\Phi}(\mathbf{x})\bar{\Theta} + \mathbf{\Phi}(\mathbf{x})\tilde{\Theta}_e \\ &= (A + B)\mathbf{x} + \frac{\partial \mathbf{\Phi}(\mathbf{0})\tilde{\Theta}_e}{\partial \mathbf{x}}\mathbf{x} + G_1(\mathbf{x}, \hat{\Theta}_e) + \mathbf{\Phi}(\mathbf{x})\bar{\Theta} \\ &= \left(A + B + \frac{\partial \mathbf{\Phi}(\mathbf{0})\tilde{\Theta}_e}{\partial \mathbf{x}}\right)\mathbf{x} + G(\mathbf{x}, \bar{\Theta}) \end{aligned}$$

where $G(\mathbf{x}, \bar{\Theta}) \triangleq G_1(\mathbf{x}, \hat{\Theta}_e) + \mathbf{\Phi}(\mathbf{x})\bar{\Theta}$. And we have

$$\dot{\bar{\Theta}} = -\mathbf{\Phi}^{\mathbf{T}}(\mathbf{x})\mathbf{x} \triangleq H(\mathbf{x}, \bar{\Theta}) \tag{22}$$

$G(\mathbf{x}, \bar{\Theta})$ satisfies

$$G(\mathbf{0}, \bar{\Theta}) = G_1(\mathbf{0}, \hat{\Theta}_e) + \mathbf{\Phi}(\mathbf{0})\bar{\Theta} = \mathbf{0} \tag{23}$$

$$\left.\frac{\partial G(\mathbf{x}, \bar{\Theta})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}, \bar{\Theta}=\mathbf{0}} = \frac{\partial G_1(\mathbf{0}, \hat{\Theta}_e)}{\partial \mathbf{x}} + \frac{\partial \mathbf{\Phi}(\mathbf{0})}{\partial \mathbf{x}}\mathbf{0} = \mathbf{0} \tag{24}$$

$$\left.\frac{\partial G(\mathbf{x}, \bar{\Theta})}{\partial \bar{\Theta}}\right|_{\mathbf{x}=\mathbf{0}} = \frac{\partial G_1(\mathbf{0}, \hat{\Theta}_e)}{\partial \bar{\Theta}} + \mathbf{\Phi}(\mathbf{0}) = \mathbf{0} \tag{25}$$

we then have

$$G(\mathbf{0}, \bar{\Theta}) = \mathbf{0}, \frac{\partial G(\mathbf{0}, \mathbf{0})}{\partial \mathbf{x}} = \mathbf{0}, \frac{\partial G(\mathbf{0}, \bar{\Theta})}{\partial \bar{\Theta}} = \mathbf{0} \tag{26}$$

And it is easy to see that

$$H(\mathbf{0}, \bar{\mathbf{\Theta}}) = \mathbf{0}, \ \frac{\partial H(\mathbf{0}, \bar{\mathbf{\Theta}})}{\partial \mathbf{x}} = \mathbf{0}, \ \frac{\partial H(\mathbf{0}, \bar{\mathbf{\Theta}})}{\partial \bar{\mathbf{\Theta}}} = \mathbf{0} \tag{27}$$

Assume that all eigenvalues of $(A + B + \frac{\partial \mathbf{\Phi}(\mathbf{0})\tilde{\mathbf{\Theta}}_e}{\partial \mathbf{x}})$ is in the open left hand plane. Since the equilibrium manifold $\mathbf{x} = h(\bar{\mathbf{\Theta}}) = \mathbf{0}$ is invariant and $\frac{\partial h(\mathbf{0})}{\partial \mathbf{\Theta}} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ is a center manifold [19]. The reduced system of Eqs. (21) and (22)

$$\dot{\bar{\Theta}} = H(\mathbf{0}, \bar{\mathbf{\Theta}}) = \mathbf{0} \tag{28}$$

is stable. By the center manifold theorem [19], the equilibrium point $(\mathbf{x}, \bar{\mathbf{\Theta}}) = (\mathbf{0}, \mathbf{0})$ is stable. The stability is global in view of Theorem 1. Then the first conclusion is proved.

Next, we assume that at least one of the eigenvalues of $(A + B + \frac{\partial \mathbf{\Phi}(\mathbf{0})\tilde{\mathbf{\Theta}}_e}{\partial \mathbf{x}})$ is in the open right hand plane. Then we linearize the system in Eqs. (21) and (22) in the vicinity of $(\mathbf{x}, \bar{\mathbf{\Theta}}) = (\mathbf{0}, \mathbf{0})$ as

$$\begin{bmatrix} \delta\dot{\mathbf{x}} \\ \delta\dot{\bar{\mathbf{\Theta}}} \end{bmatrix} = \begin{bmatrix} A + B + \frac{\partial \mathbf{\Phi}(\mathbf{0})\tilde{\mathbf{\Theta}}_e}{\partial \mathbf{x}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\bar{\mathbf{\Theta}} \end{bmatrix} \tag{29}$$

In view of the linearization theorem, the second conclusion is proved. ■

Definition. A function $\mathbf{f}\colon D \mapsto R^n$ is *uniformly increasing* in some convex set $D \subset R^n$ if there exists $\alpha > 0$ such that for all $\mathbf{x}, \mathbf{x}' \in D$

$$(\mathbf{x} - \mathbf{x}')^T(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}')) \geq \alpha||\mathbf{x} - \mathbf{x}'||^2 \tag{30}$$

The following theorem is useful in this paper.

THEOREM 4 ([10]) *A function* $\mathbf{f} \in C^1[R^n, R^n]$ *is uniformly increasing in* R^n *if and only if for some* $\alpha > 0$ *such that* $(D\mathbf{f}(\mathbf{x}) - \alpha\mathbf{I})$ *is positive definite for all* $\mathbf{x} \in R^n$*, where* $\mathbf{I}$ *is the* $n \times n$ *identity matrix.* $D\mathbf{f}(\mathbf{x})$ *denotes the Jacobian matrix of* $\mathbf{f}(\mathbf{x})$.

Then we have the following theorem

THEOREM 5 *The equilibrium point* $(\mathbf{x}, \hat{\mathbf{\Theta}}) = (\mathbf{0}, \hat{\mathbf{\Theta}}_e)$ *is globally stable if* $-\mathbf{\Phi}(\mathbf{x})\tilde{\mathbf{\Theta}}_e$ *is uniformly increasing in* R^n *such that*

$$\alpha\mathbf{I} + (A + B) + D\mathbf{\Phi}(\mathbf{x})\tilde{\mathbf{\Theta}}_e \tag{31}$$

is negative definite for all $\mathbf{x} \in R^n$.

Proof: Let

$$\bar{\mathbf{\Theta}} = \hat{\mathbf{\Theta}}_e - \hat{\mathbf{\Theta}} = \tilde{\mathbf{\Theta}} - \tilde{\mathbf{\Theta}}_e \tag{32}$$

Then we have

$$\begin{aligned}\dot{\mathbf{x}} &= (A+B)\mathbf{x} + \mathbf{\Phi}(\mathbf{x})\tilde{\mathbf{\Theta}} \\ &= (A+B)\mathbf{x} + \mathbf{\Phi}(\mathbf{x})(\bar{\mathbf{\Theta}} + \tilde{\mathbf{\Theta}}_e) \\ &= (A+B)\mathbf{x} + \mathbf{\Phi}(\mathbf{x})\tilde{\mathbf{\Theta}}_e + \mathbf{\Phi}(\mathbf{x})\bar{\mathbf{\Theta}}\end{aligned} \tag{33}$$

$$\dot{\bar{\Theta}} = -\mathbf{\Phi}^{\mathbf{T}}(\mathbf{x})\mathbf{x} \tag{34}$$

Then we construct the Lyapunov function

$$V(\mathbf{x}, \bar{\mathbf{\Theta}}) = \frac{1}{2}\mathbf{x}^T\mathbf{x} + \frac{1}{2}\bar{\mathbf{\Theta}}^T\bar{\mathbf{\Theta}} \tag{35}$$

Differentiating V along the trajectories of solutions of Eqs. (33) and (34), we get

$$\begin{aligned}\dot{V}(\mathbf{x}, \bar{\mathbf{\Theta}}) &= \mathbf{x}^T(A+B)\mathbf{x} + \mathbf{x}^T\mathbf{\Phi}(\mathbf{x})\tilde{\mathbf{\Theta}}_e + \mathbf{x}^T\mathbf{\Phi}(\mathbf{x})\bar{\mathbf{\Theta}} - \bar{\mathbf{\Theta}}^T\mathbf{\Phi}^T(\mathbf{x})\mathbf{x} \\ &= \mathbf{x}^T(A+B)\mathbf{x} + \mathbf{x}^T(\mathbf{\Phi}(\mathbf{x}) - \mathbf{\Phi}(\mathbf{0}))\tilde{\mathbf{\Theta}}_e \\ &\leq -\alpha\mathbf{x}^T\mathbf{x}\end{aligned} \tag{36}$$

■

In some cases, we are also interested in controlling a chaotic system to a non-zero point, then the following corollary is useful.

COROLLARY 1 *Assume that* $\mathbf{z} = \mathbf{x} - \mathbf{x}_e$, *where* $\mathbf{x}_e$ *is a non-zero equilibrium point, with the control law*

$$\mathbf{u} = -A\mathbf{x}_e + B\mathbf{z} - \mathbf{\Phi}(\mathbf{x})\hat{\mathbf{\Theta}} \tag{37}$$

and the parameter updating law

$$\dot{\hat{\mathbf{\Theta}}} = \mathbf{\Phi}^T(\mathbf{x})\mathbf{z} \tag{38}$$

then the close-loop adaptive system has a globally stable equilibrium point $(\mathbf{x}, \hat{\mathbf{\Theta}}) = (\mathbf{x}_e, \mathbf{\Theta})$. *Furthermore,* $\lim_{t\longrightarrow\infty}\mathbf{z}(\mathbf{t}) = \mathbf{0}$.

Proof:

$$\begin{aligned}\dot{\mathbf{z}}(\mathbf{t}) &= A(\mathbf{z} + \mathbf{x}_e) + \mathbf{\Phi}(\mathbf{x})\mathbf{\Theta} + (-A\mathbf{x}_e + B\mathbf{z} - \mathbf{\Phi}(\mathbf{x})\hat{\mathbf{\Theta}}) \\ &= (A+B)\mathbf{z} + \mathbf{\Phi}(\mathbf{x})\tilde{\mathbf{\Theta}}\end{aligned} \tag{39}$$

In view of Theorem 1, we finish the proof. ■

3. Examples of Control of Chua's Circuit with Cubic Nonlinearity

The well-known chaotic circuit model of Chua's circuit [15] has a piece-wise linear element called Chua's diode [16]. Recently it was found that not all features of a real circuit are captured correctly by this piecewise-linear circuit [17]. In this paper, we control one kind of Chua's circuit with smooth nonlinearity called Chua's circuit with cubic nonlinearity which is given by [18]

$$\begin{cases} \dot{x}_1 = px_2 + a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -qx_2 \end{cases} \tag{40}$$

Example 1. Control Chua's circuit to the origin.

In this example, we choose the unknown parameter vector as [18]

$$\mathbf{\Theta} = \begin{pmatrix} p \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ q \end{pmatrix} \tag{41}$$

Then the Chua's circuit in Eq. (40) can be decomposed as

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} + \underbrace{\begin{pmatrix} x_2 & 1 & x_1 & x_1^2 & x_1^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_2 \end{pmatrix}}_{\mathbf{\Phi}(\mathbf{x})} \begin{pmatrix} p \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ q \end{pmatrix} + \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}}_{\mathbf{u}} \tag{42}$$

then we choose

$$B = \begin{pmatrix} -2 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \tag{43}$$

such that

$$A + B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{44}$$

is negative definite. The control law is given by

$$\mathbf{u} = \begin{pmatrix} -2 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} x_2 & 1 & x_1 & x_1^2 & x_1^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_2 \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{a}_0 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{q} \end{pmatrix}$$

$$= \begin{pmatrix} -\hat{p}x_2 - \hat{a}_0 - (2+\hat{a}_1)x_1 - \hat{a}_2 x_1^2 - \hat{a}_3 x_1^3 \\ -x_1 - x_3 \\ \hat{q}x_2 - x_3 \end{pmatrix} \tag{45}$$

The parameter update law is given by

$$\begin{pmatrix} \dot{\hat{p}} \\ \dot{\hat{a}}_0 \\ \dot{\hat{a}}_1 \\ \dot{\hat{a}}_2 \\ \dot{\hat{a}}_3 \\ \dot{\hat{q}} \end{pmatrix} = \begin{pmatrix} x_2 & 0 & 0 \\ 1 & 0 & 0 \\ x_1 & 0 & 0 \\ x_1^2 & 0 & 0 \\ x_1^3 & 0 & 0 \\ 0 & 0 & -x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ x_1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \\ -x_2 x_3 \end{pmatrix} \tag{46}$$

The control law provided in Theorem 1 can guarantee that the controlled Chua's circuit asymptotically approach the origin. The linear system

$$\dot{\mathbf{x}} = (A+B)\mathbf{x} \tag{47}$$

is the dynamics in the $\mathbf{x}$-subspace of the close loop system. Assume that the parameter vector of the Chua's circuit is given by

$$\mathbf{\Theta} = \begin{pmatrix} p \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ q \end{pmatrix} = \begin{pmatrix} 10 \\ 0.1 \\ \frac{10}{7} \\ 0.1 \\ -\frac{20}{7} \\ \frac{100}{7} \end{pmatrix} \tag{48}$$

The initial condition of the Chua's circuit is given by

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0.442006 \\ -0.213984 \\ -0.909130 \end{pmatrix} \tag{49}$$

The initial condition of the parameter update law is given by

$$\begin{pmatrix} \hat{p}(0) \\ \hat{a}_0(0) \\ \hat{a}_1(0) \\ \hat{a}_2(0) \\ \hat{a}_3(0) \\ \hat{q}(0) \end{pmatrix} = 2 \begin{pmatrix} p \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ q \end{pmatrix} \tag{50}$$

After controlling, the Chua's circuit asymptotically approaches the origin as the solid lines shown in Fig. 1(a). The dashed lines show the dynamics of the uncontrolled Chua's circuit. The dynamics of $\hat{\mathbf{\Theta}}$ is shown in Fig. 1(b)–(g), one can see that $\hat{\mathbf{\Theta}}$ approaches a constant vector but not $\mathbf{\Theta}$.

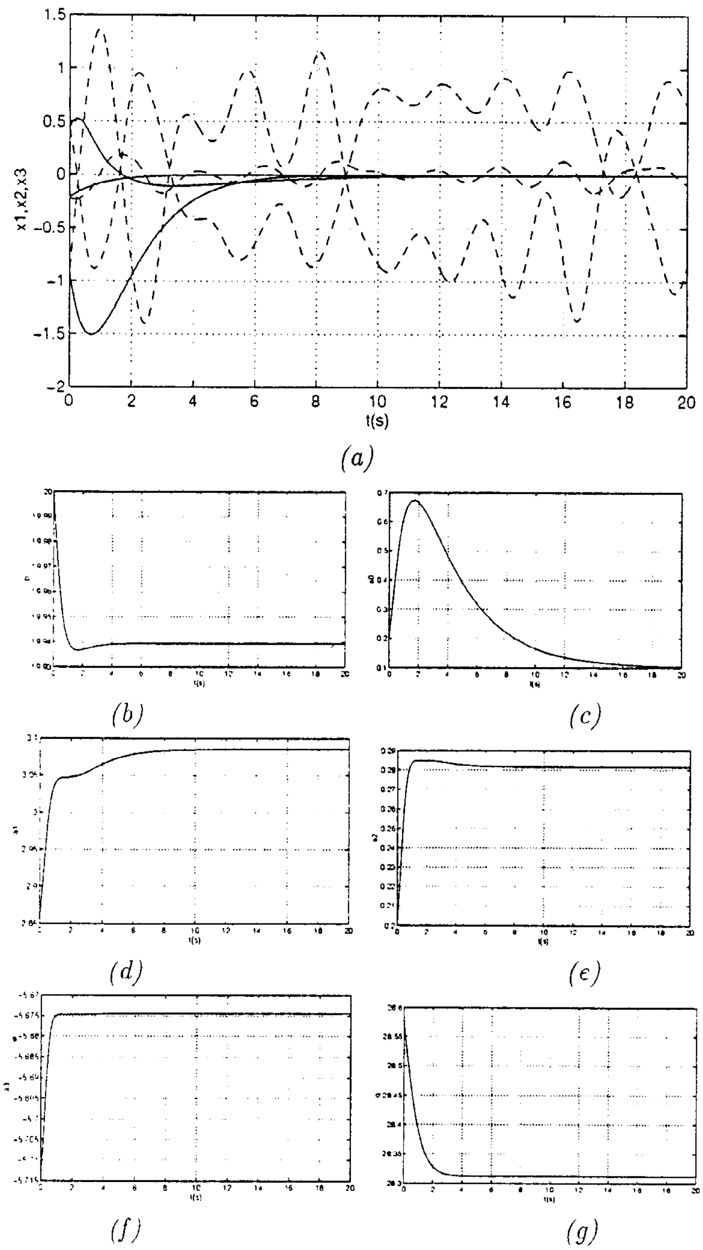

Figure 1. Control a Chua's circuit with 6 unknown parameters to the origin. (a) The uncontrolled state variables (dashed lines) and controlled state variables (solid lines). (b) The update process of $\hat{p}$. (c) The update process of $\hat{a}_0$. (d) The update process of $\hat{a}_1$. (e) The update process of $\hat{a}_2$. (f) The update process of $\hat{a}_3$. (g) The update process of $\hat{q}$.

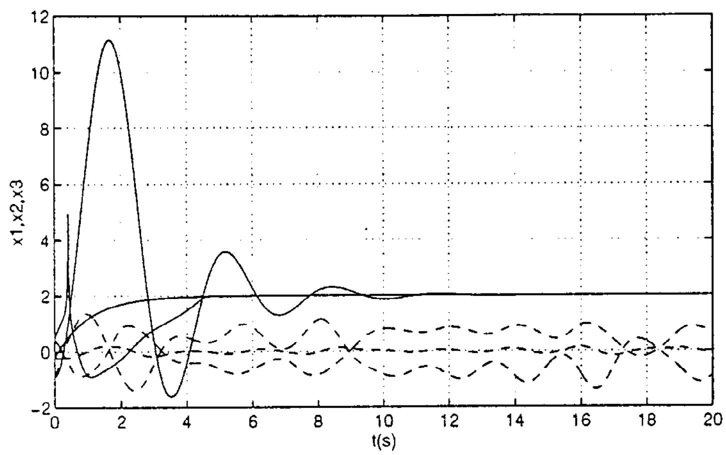

Figure 2. Control a Chua's circuit with 6 unknown parameters to a point $\mathbf{x}_e$. The uncontrolled state variables (dashed lines) and controlled state variables (solid lines) are shown.

Example 2. Control Chua's circuit to $\mathbf{x}_e \neq \mathbf{0}$.

We then control Chua's circuit to a point $\mathbf{x}_e \neq \mathbf{0}$. By using corollary 1, we choose the control law as

$$\mathbf{u} = \begin{pmatrix} -\hat{p}x_2 - \hat{a}_0 - (2 + \hat{a}_1)x_1 - \hat{a}_2 x_1^2 - \hat{a}_3 x_1^3 + 2x_{1e} \\ -x_1 - x_3 + x_{2e} \\ \hat{q}x_2 - x_3 + x_{3e} \end{pmatrix} \tag{51}$$

The parameter update law is given by

$$\begin{pmatrix} \hat{p} \\ \hat{a}_0 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{q} \end{pmatrix} = \begin{pmatrix} x_1 x_2 - x_{1e} x_2 \\ x_1 - x_{1e} \\ x_1^2 - x_{1e} x_1 \\ x_1^3 - x_{1e} x_1^2 \\ x_1^4 - x_{1e} x_1^3 \\ -x_2(x_3 - x_{3e}) \end{pmatrix} \tag{52}$$

The initial conditions for Chua's circuit and the parameter update law are the same as those in Example 1. And we choose $\mathbf{x}_e$ as

$$\begin{pmatrix} x_{1e} \\ x_{2e} \\ x_{3e} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \tag{53}$$

Fig. 2 shows the simulation result. We can see that the Chua's circuit asymptotically approaches $\mathbf{x}_e$.

4. Conclusions

We present an adaptive controller for stabilizing chaotic systems with unknown parameters to fixed points. We use a Lyapunov function based method to design the parameter estimator such that the controlled chaotic system can be globally stabilized to the reference point. We also give a detailed study of the dynamics of the parameter estimator. We use the center manifold theory to study the stability of the parameter estimator.

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