Suppose that $(\mathbf{A}, \mathbf{b})$ and $(\tilde{\mathbf{A}}, \tilde{\mathbf{b}})$ are controllable. Then system (18) and system (19) are linearly conjugate if the eigenvalues of $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are the same and the eigenvalues of $\mathbf{A} + \mathbf{b}\mathbf{w}_i^T$ and $\tilde{\mathbf{A}} + \tilde{\mathbf{b}}\tilde{\mathbf{w}}_i^T$ are the same for each i.

Proof: Let $\mathbf{K} = \mathbf{K}(\mathbf{A}^T, \mathbf{b}^T)$ and $\tilde{\mathbf{K}} = \mathbf{K}(\tilde{\mathbf{A}}^T, \tilde{\mathbf{b}}^T)$, which by hypothesis are nonsingular. Using the transformations $\mathbf{x} = \mathbf{K}^T\mathbf{y}$, and $\tilde{\mathbf{x}} = \tilde{\mathbf{K}}^T\tilde{\mathbf{y}}$, we get the following systems.

$$\begin{aligned}\dot{\mathbf{y}} &= (\mathbf{K}^T)^{-1}\mathbf{A}\mathbf{K}^T\mathbf{y} \\ &\quad + f\Big(\mathbf{w}_1^T\mathbf{K}^T\mathbf{y}, \mathbf{w}_2^T\mathbf{K}^T\mathbf{y}, \cdots, \mathbf{w}_i^T\mathbf{K}^T\mathbf{y}, \cdots\Big)(\mathbf{K}^T)^{-1}\mathbf{b} \\ &= \hat{\mathbf{A}}^T\mathbf{y}f\Big(\mathbf{w}_1^T\mathbf{K}^T\mathbf{y}, \mathbf{w}_2^T\mathbf{K}^T\mathbf{y}, \cdots, \mathbf{w}_i^T\mathbf{K}^T\mathbf{y}, \cdots\Big)\mathbf{e}_1 \\ \dot{\tilde{\mathbf{y}}} &= (\tilde{\mathbf{K}}^T)^{-1}\tilde{\mathbf{A}}\tilde{\mathbf{K}}^T\tilde{\mathbf{y}} \\ &\quad + f\Big(\tilde{\mathbf{w}}_1^T\tilde{\mathbf{K}}^T\tilde{\mathbf{y}}, \tilde{\mathbf{w}}_2^T\tilde{\mathbf{K}}^T\tilde{\mathbf{y}}, \cdots, \tilde{\mathbf{w}}_i^T\tilde{\mathbf{K}}^T\tilde{\mathbf{y}}, \cdots\Big)(\tilde{\mathbf{K}}^T)^{-1}\tilde{\mathbf{b}} \\ &= \hat{\mathbf{A}}^T\tilde{\mathbf{y}} + f\Big(\tilde{\mathbf{w}}_1^T\tilde{\mathbf{K}}^T\tilde{\mathbf{y}}, \tilde{\mathbf{w}}_2^T\tilde{\mathbf{K}}^T\tilde{\mathbf{y}}, \cdots, \tilde{\mathbf{w}}_i^T\tilde{\mathbf{K}}^T\tilde{\mathbf{y}}, \cdots\Big)\mathbf{e}_1\end{aligned}$$

where $\hat{\mathbf{A}}$ is defined in (2). By Lemma 3 and the hypothesis, for each i, the vectors $\mathbf{K}\mathbf{w}_i$ and $\tilde{\mathbf{K}}\tilde{\mathbf{w}}_i$ are uniquely determined by the eigenvalues of $\mathbf{A}$ and $\mathbf{A} + \mathbf{b}\mathbf{w}_i^T$ and are thus equal to each other. ■

IV. Conclusion

In this letter we showed how Lur'e type systems are linearly conjugate whenever the scalar nonlinearities and certain sets of eigenvalues are matched. Furthermore, two Lur'e type systems are linearly conjugate if the equilibrium points are matched and the eigenvalues of the Jacobian matrices are matched. This implies that almost all continuous piecewise-linear vector fields with parallel boundary planes are linearly conjugate if the boundary planes, equilibrium points and eigenvalues in corresponding regions are matched.

Note Added in Proof: Theorem 1 has also been proved in [5].

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On a Conjecture Regarding the Synchronization in an Array of Linearly Coupled Dynamical Systems

Chai Wah Wu and Leon O. Chua

Abstract—**In this letter, we give supporting evidence for a conjecture regarding the amount of coupling needed to synchronize an array of linearly coupled dynamical systems. Roughly speaking, the conjecture says that the coupling needed to synchronize an array of coupled systems is inversely proportional to the nonzero eigenvalue of the coupling graph that is smallest in magnitude. The conjecture implies that the coupling needed to synchronize an array can be derived from the coupling topology and the coupling needed to synchronize two coupled dynamical systems.**

I. Introduction

In [1], a sufficient condition for synchronization in an array of linearly coupled identical dynamical systems is obtained. This sufficient condition is related to the eigenvalues of the coupling matrices. It was conjectured in [1] that the minimal coupling needed also follows the same asymptotic behavior as the number of cells goes to infinity. In this letter, we propose a more general and precise statement of this conjecture and provide some evidence for it. Before we state the conjecture, let us give some motivation using an example.

Consider m identical Chua's oscillators connected in a chain (Fig. 1), where G_c is the value of the coupling conductance. Calculating $G_c = G_{min}$, the minimum amount of coupling conductance needed to completely synchronize the array, we find that G_{min} increases with the number of Chua's oscillators m. Fig. 2 shows the graph of $\frac{1}{G_{min}}$ versus m.

The graph corresponding to Fig. 1 is the path graph and has a Laplacian matrix

$$\mathbf{L} = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & -1 & 1 \end{pmatrix}.$$

The smallest *nonzero* eigenvalue of $\mathbf{L}$ is equal to $4\sin^2\left(\frac{\pi}{m}\right)$, which we plot in Fig. 3. We notice that Fig. 3 is similar to Fig. 2. In fact, the conjecture states that except for a constant factor, these two graphs are identical. The purpose of this letter is to give supporting evidence for this conjecture based on numerical simulations.

II. Synchronization in an Array of Linearly Coupled Dynamical Systems

The general framework we consider is an array of coupled identical systems, consisting of m cells, each cell being an n-dimensional system:

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1, t) \\ \vdots \\ \mathbf{f}(\mathbf{x}_m, t) \end{pmatrix} + \alpha(\mathbf{G} \otimes \mathbf{D})\mathbf{x} \tag{1}$$

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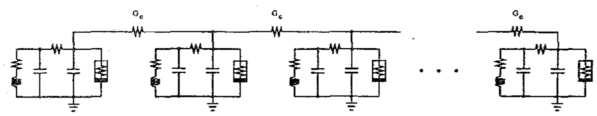

Fig. 1. An array of identical Chua's oscillators arranged in a chain. G_c is the conductance of the coupling resistors.

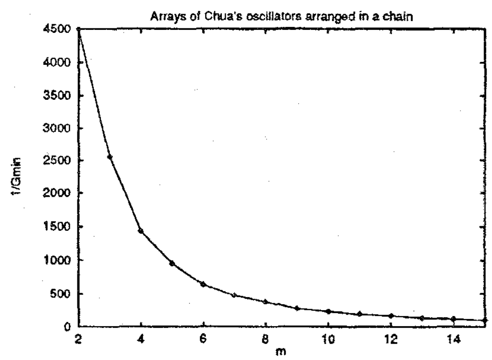

Fig. 2. Plot of $\frac{1}{G_{min}}$ versus m for an array of Chua's oscillators arranged in a chain.

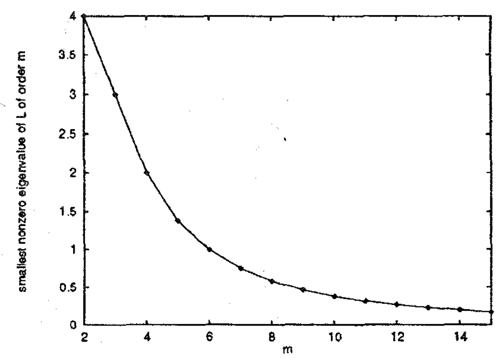

Fig. 3. Smallest nonzero eigenvalue of $\mathbf{L}$ as a function of m.

where $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)^T$, $\mathbf{x}_i \in \mathbb{R}^n$, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\mathbf{G}$ is an $m \times m$ *real-valued* matrix which depends on the coupling topology of the array, $\mathbf{D}$ is an $n \times n$ *real-valued* matrix and $\otimes$ denotes the Kronecker product operator. The real scalar-valued parameter α is used to change the coupling strength.

We propose the following conjecture for the coupling required for synchronization:

Conjecture 1: Consider two arrays of coupled systems of m_1 and m_2 cells, respectively.

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1, t) \\ \vdots \\ \mathbf{f}(\mathbf{x}_{m_1}, t) \end{pmatrix} + \alpha_1(\mathbf{G}_1 \otimes \mathbf{D})\mathbf{x} \tag{2}$$

$$\dot{\tilde{\mathbf{x}}} = \begin{pmatrix} \mathbf{f}(\tilde{\mathbf{x}}_1, t) \\ \vdots \\ \mathbf{f}(\tilde{\mathbf{x}}_{m_2}, t) \end{pmatrix} + \alpha_2(\mathbf{G}_2 \otimes \mathbf{D})\tilde{\mathbf{x}} \tag{3}$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are $m_1 \times m_1$ and $m_2 \times m_2$ matrices, respectively, $(m_1, m_2 \geq 2)$ and α_1, α_2 are real numbers. Assume that $\mathbf{G}_1$ and $\mathbf{G}_2$ are symmetric, have zero row sums, and contain only nonpositive eigenvalues such that 0 is an eigenvalue of multiplicity 1. Let μ_1 and μ_2 be the least negative *nonzero* eigenvalues of $\mathbf{G}_1$ and $\mathbf{G}_2$, respectively. Suppose μ_1 and μ_2 are related as follows:

$$\mu_1 \times \alpha_1 = \mu_2 \times \alpha_2. \tag{4}$$

Then array (2) globally synchronizes[1] if and only if array (3) globally synchronizes.

Another way to state this conjecture is that if array (2) globally synchronizes for some α_1, then array (3) globally synchronizes for $\alpha_2 = \frac{\mu_1\alpha_1}{\mu_2}$. Note that the assumptions on $\mathbf{G}_1$ and $\mathbf{G}_2$ are satisfied if $\mathbf{G}_1$ and $\mathbf{G}_2$ are zero row sum matrices which are symmetric and irreducible, and have nonnegative off-diagonal elements, as is the case with many coupling configurations studied in the literature [2], [3], [4], [1].

The implication of this conjecture is that synchronization in an array of cells can be determined from the synchronization in *two* coupled cells.

The purpose of this paper is to give supporting evidence for this conjecture when the cell is Chua's oscillator.

In order to deduce that the array of Chua's oscillators is synchronized, we pick a random initial condition, simulate the system for some time in order for the transients to die down, then calculate the synchronization error as defined by

$$\sqrt{\frac{1}{(m-1)n_t}\sum_{i=1}^{n_t}\sum_{j=1}^{m-1}||\mathbf{x}_j(t_i) - \mathbf{x}_{j+1}(t_i)||^2}$$

where $t_1, \ldots t_{n_t}$ are times after the transient time. If the synchronization error is smaller than some threshold ϵ, then the array is considered synchronized.

Using this criteria, we calculate the minimal coupling (by varying α in (1)) needed to synchronize the array, and if Conjecture 1 is true, then these minimal coupling should be related by (4). In other words, $\mu \times \alpha$ should be a constant, where α is the minimal coupling coefficient needed to synchronize the array (1) and μ is the nonzero eigenvalue of the coupling matrix $\mathbf{G}$ that is smallest in magnitude.

In the simulations, we fix the following values for $\mathbf{f}$ and $\mathbf{D}$:

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}\begin{pmatrix} v_1 \\ v_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{C_1}\left(\frac{v_2 - v_1}{R} - h(v_1)\right) \\ \frac{1}{C_2}\left(\frac{v_1 - v_2}{R} + i_3\right) \\ -\frac{1}{L}(v_2 + R_0 i_3) \end{pmatrix} \tag{5}$$

where $\mathbf{x} = (v_1, v_2, i_3)^T$ and

$$\left.\begin{array}{lcl} C_1 & = & 10 \times 10^{-9} \\ C_2 & = & 100 \times 10^{-9} \\ L & = & 18 \times 10^{-3} \\ R & = & 1600 \\ R_0 & = & 0 \\ h(v_1) & = & -0.409 \times 10^{-3} v_1 \\ & & \quad -0.1735 \times 10^{-3}\{|v_1 + 1| - |v_1 - 1|\} \end{array}\right\} \tag{6}$$

and

$$\mathbf{D} = \begin{pmatrix} \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

We choose $\mathbf{G}$ for various m and various topologies and calculate the corresponding values of $-\frac{1}{\mu\alpha}$.

[1] i.e., $||\mathbf{x}_i - \mathbf{x}_j|| \to 0$ as $t \to \infty$ for all i, j and all initial conditions.

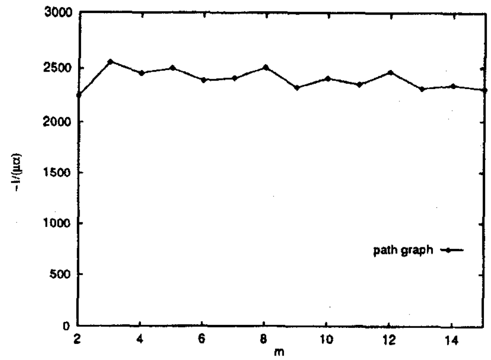

Fig. 4. $-\frac{1}{\mu\alpha}$ as a function of m for the path graph configuration.

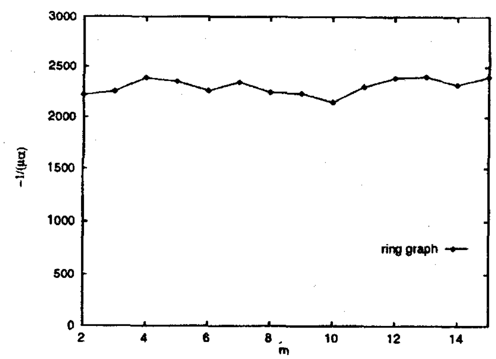

Fig. 5. $-\frac{1}{\mu\alpha}$ as a function of m for the ring graph configuration.

2.1 Path graph configuration

First we choose Chua's oscillators arranged in a path graph (Fig. 1). The corresponding $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ & & & 1 & -1 \end{pmatrix}$$

The simulation results are shown in Fig. 4.

We see that the value of $-\frac{1}{\mu\alpha}$ is relatively constant as m is varied and centers around 2300.

2.2 Ring graph configuration

The matrix $\mathbf{G}$ corresponding to Chua's oscillator in a ring graph configuration is given by:

$$\mathbf{G} = \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ 1 & & & 1 & -2 \end{pmatrix}.$$

The simulation results are shown in Fig. 5

2.3 Star Graph Configuration

We denote $K_{i,j}$ as the bipartite graph with i and j vertices in the two partitions, respectively. We denote K_m as the complete graph of m vertices.

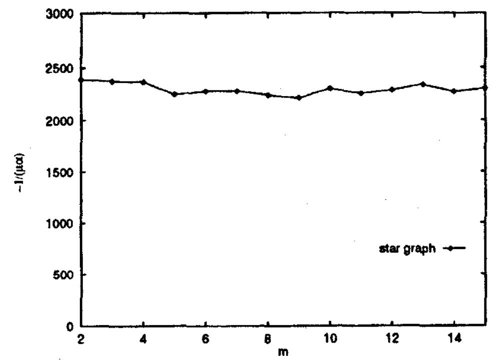

Fig. 6. $-\frac{1}{\mu\alpha}$ as a function of m for the star graph ($K_{1,m-1}$) configuration.

Next we choose Chua's oscillators arranged in a star graph, or $K_{1,m-1}$. The corresponding $\mathbf{G}$ is given by:

$$\mathbf{G} = \begin{pmatrix} -m+1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & & & \\ 1 & & -1 & & \\ 1 & & & \ddots & \\ 1 & & & & -1 \end{pmatrix}.$$

The simulation results are shown in Fig. 6.

2.4 Bipartitite Graph Configuration

In Fig. 7 simulation results are shown for Chua's oscillators arranged in the graph $K_{\lfloor m/2\rfloor,\lfloor m/2\rfloor+1}$ if m is odd and the graph $K_{m/2,m/2}$ if m is even. The corresponding $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{pmatrix} -\lfloor m/2\rfloor-1 & & & 1 & \cdots & 1 \\ & \ddots & & 1 & \cdots & 1 \\ & & -\lfloor m/2\rfloor-1 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & -\lfloor m/2\rfloor & & \\ 1 & \cdots & 1 & & \ddots & \\ 1 & \cdots & 1 & & & -\lfloor m/2\rfloor \end{pmatrix}$$

when m is odd and $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{pmatrix} -m/2 & & & 1 & \cdots & 1 \\ & \ddots & & 1 & \cdots & 1 \\ & & -m/2 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & -m/2 & & \\ 1 & \cdots & 1 & & \ddots & \\ 1 & \cdots & 1 & & & -m/2 \end{pmatrix}$$

when m is even.

2.5 Fully Connected Configuration

Next we choose Chua's oscillators arranged in a fully connected graph K_m. The corresponding matrix $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{pmatrix} -m+1 & 1 & \cdots & & 1 \\ 1 & -m+1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & 1 & 1 \\ 1 & \cdots & & 1 & -m+1 \end{pmatrix}$$

The simulation results are shown in Fig. 8.

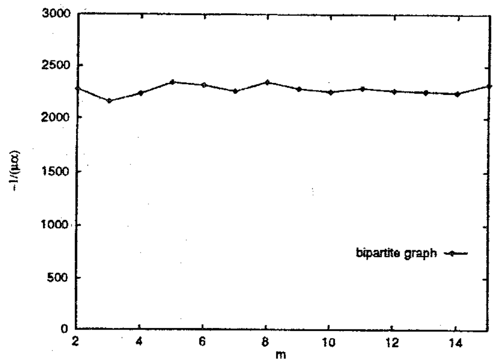

Fig. 7. $-\frac{1}{\mu\alpha}$ as a function of m for the bipartite graph configuration.

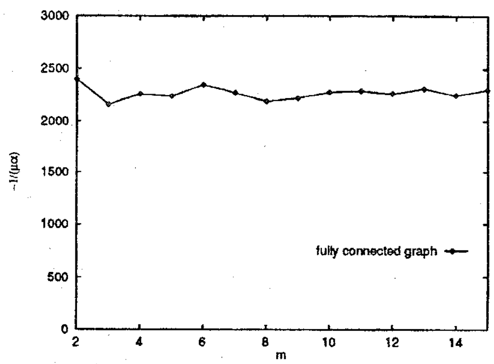

Fig. 8. $-\frac{1}{\mu\alpha}$ as a function of m for the fully connected graph configuration.

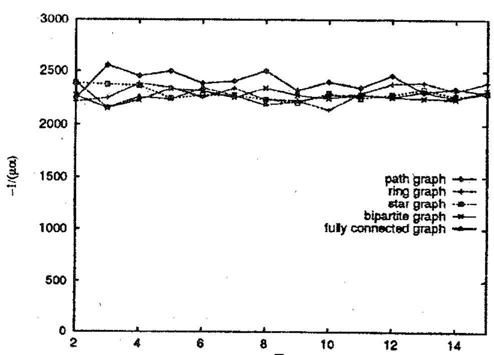

Fig. 9. Superposition of Figs. 4-8.

Superimposing the results above in Fig. 9, we see that the value of $-\frac{1}{\mu\alpha}$ is relatively close to the constant value 2300 for the different types of graphs considered.

In the previous cases $-\mathbf{G}$ is the Laplacian matrix of a graph, i.e., it is symmetric, have zero-row sums and contains only 0 and -1 in the off-diagonal elements [5]. Let us now consider more general $\mathbf{G}$'s.

In particular, we generated $\mathbf{G}$ by picking the elements above the main diagonal randomly and independently from a Gaussian distribution with variance 1 and mean 1.5. We then set the other elements such that $\mathbf{G}$ is symmetric and has zero row sums. The results are shown in Fig. 10. We see that $-\frac{1}{\mu\alpha}$ is nearly constant, but at a higher value that the previous simulation results.

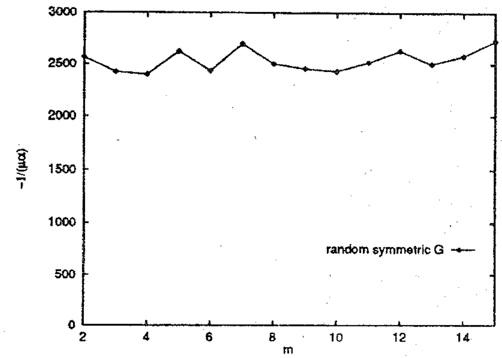

Fig. 10. $-\frac{1}{\mu\alpha}$ as a function of m for randomly chosen symmetric matrices $\mathbf{G}$.

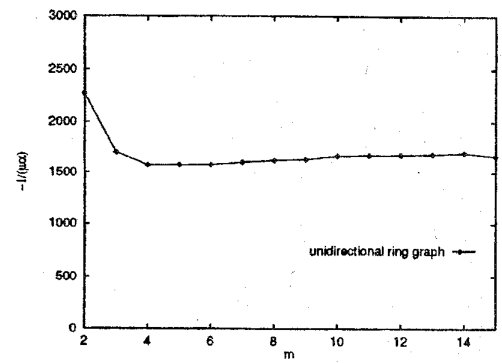

Fig. 11. $-\frac{1}{\mu\alpha}$ as a function of m for the unidirectional ring graph configuration.

III. Nonsymmetric Irreducible Coupling Matrix G

In Conjecture 1, we require the coupling matrix $\mathbf{G}$ to be symmetric, as is the case with the simulation results so far. Let us now look at some cases where the matrix is irreducible but not symmetric and see if the conclusion of the conjecture still holds. As the eigenvalues of $\mathbf{G}$ can be complex now, we will choose $-\mu$ to be the magnitude of the nonzero eigenvalue of $\mathbf{G}$ with the smallest magnitude.

3.1 Unidirectional Ring Configuration

In this configuration, the matrix $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \\ 1 & & & & -1 \end{pmatrix}.$$

The simulation results are shown in Fig. 11. We notice that $-\frac{1}{\mu\alpha}$ is nearly constant (except for the case $m = 2$), but the constant value is lower than in the cases considered in Section II.

3.2 Random Coupling Configuration

For this case, we generated $\mathbf{G}$ by picking the off-diagonal elements randomly and independently from a Gaussian distribution with variance 1 and mean 1 and setting the diagonal elements such that $\mathbf{G}$ has zero row sums. The results are shown in Fig. 12. We see larger fluctuations in $-\frac{1}{\mu\alpha}$, but $-\frac{1}{\mu\alpha}$ still centers around some value.

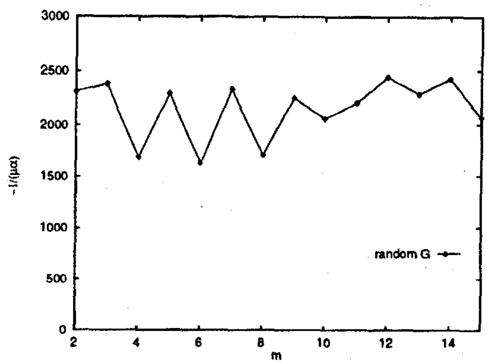

Fig. 12. $-\frac{1}{\mu\alpha}$ as a function of m for the random coupling configuration.

IV. Conclusion

Given the simulation results in Section II, we can estimate that an array of Chua's oscillators, coupled via linear resistors across the nonlinear resistors[2], will synchronize if the coupling conductance G_c is larger than $\frac{1}{2300c}$, where the parameters of Chua's oscillator are given by (5)–(6) and c is the algebraic connectivity[3] of the corresponding connectivity graph [5]. The value $\frac{1}{2300}$ was found by finding the minimal coupling conductance required to synchronize two coupled Chua's oscillators (for the given parameters) and multiplying it by 2, since 2 is the algebraic connectivity of the connected graph with two vertices.

In general, we can conclude that the minimal coupling coefficient α follows the rule dictated by the conjecture fairly well, especially when $\mathbf{G}$ is a symmetric matrix, has zero row sums and contains only 0 and 1 in the off-diagonal elements. For more general $\mathbf{G}$ with zero row sums, there is a larger fluctuation between the computed and the predicted values. This case requires further research. The conjecture allows us to predict the synchronization of an array of cells based on the synchronization properties of two coupled cells. We have only verified this conjecture using arrays of Chua's oscillators. It remains to be seen whether this conjecture is true in general, for other types of systems and other types of linear coupling (i.e., different $\mathbf{D}$). The motivation for this conjecture comes from synchronization results obtained using a quadratic Lyapunov function. The reason this conjecture is true for arrays of Chua's oscillators might depend on whether quadratic Lyapunov functions are optimal or near-optimal for Chua's oscillators. Further research is needed to clarify this.

[2]This corresponds to $\mathbf{D}$ as given in (7) and $\mathbf{G}$ being symmetric. Fig. 1 is an example of such coupling.

[3]The algebraic connectivity of a graph is defined as the smallest nonzero eigenvalue of the corresponding Laplacian matrix.

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