SIMPLE THREE OSCILLATOR UNIVERSAL PROBES FOR DETERMINING SYNCHRONIZATION STABILITY IN COUPLED ARRAYS OF OSCILLATORS

Chai Wah Wu

IBM Research Division, Thomas J. Watson Research Center
P. O. Box 218, Yorktown Heights, NY 10598, U.S.A.
e-mail: chaiwah@watson.ibm.com

ABSTRACT

In [1], Fink et al. describe how a coupled array of three oscillators can serve as a universal probe of the synchronization properties in general coupled arrays of oscillators. However, the coupling topology in this probe can cause difficulties in experimental implementations. For instance, every oscillator is coupled to every other oscillator resulting in many coupling terms. Furthermore, the variable coupling weights at several coupling terms need to be set equal to each other. This means that some component values need to be varied over a range and be perfectly matched to each other in the experimental setup. We present simpler universal probes by utilizing the minimal number of coupling terms and by minimizing the constraint of having the coupling weights matched.

1. INTRODUCTION

In [1], Fink et al. describe how a coupled array of three oscillators can serve as a universal probe of the synchronization properties in general coupled arrays of oscillators. The universal probe allows the synchronization stability in arbitrary coupled arrays of oscillators to be determined by observing whether the universal probe synchronizes for a discrete set of parameters. However, the coupling topology in this probe can cause difficulties in experimental implementations. For instance, every oscillator is coupled to every other oscillator. As more coupling terms generally imply increased complexity, a minimal number of coupling terms is preferred. Furthermore, the coupling weights at several coupling terms need to be set equal to each other. This means that some component values need to be varied over a range and at the same time be perfectly matched to each other in the experimental setup. We present simpler universal probes by utilizing the minimal number of coupling terms and by minimizing the constraint of having the coupling weights matched.

2. MASTER STABILITY FUNCTION

Consider the coupled array of oscillators:

$$\dot{x} = I \odot f(x_i) + G \odot H(x_i)$$  (1)

where $G$ is an irreducible matrix with zero row sums, $x = (x_1, x_2, \ldots)^T$ and we use the notation

$$A \odot B(x_i) = \begin{pmatrix} A_{11}B(x_1) & A_{12}B(x_2) & \cdots & A_{1n}B(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B(x_1) & A_{m2}B(x_2) & \cdots & A_{mn}B(x_n) \end{pmatrix}$$

As noted in [2, 3], the case where $G$ has a constant row sum $\gamma$ can be reduced to the case of $G$ having zero row sums by replacing $f(x_i)$ with $f(x_i) + \gamma H(x_i)$.

In [4], it is shown that the Lyapunov exponents transverse to the synchronization manifold of the coupled array (1) can be deduced from the Lyapunov exponents of the variational equations:

$$\dot{\eta} = (Df(x) + \mu DH(x)) \eta$$  (2)

for each $\mu$ a nonzero eigenvalue of $G$. This leads to the study of a three oscillator probe in [1], which contains variational equations of the form (2) for some $\mu$. By varying two real parameters, $\mu$ can be set to any complex number. Thus this three oscillator probe can determine the transverse Lyapunov exponents, and hence the synchronization stability of the coupled array, by setting the parameter $\mu$ to each of the nonzero eigenvalues of $G$.

The three oscillator probe has the state equations (1) with coupling matrix

$$G = \begin{pmatrix} -2\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & -2\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -2\frac{\sqrt{3}}{3} \end{pmatrix}$$  (3)

$G$ describes how the three oscillators are coupled to each other. The variational equation contains Eq. (2) with $\mu = \epsilon + i\delta$.  

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In practical implementations, the probe (1) with $G$ defined as (3) can cause some difficulties. First of all, the form of $G$ implies that all three oscillators are connected to each other. There are 6 nonzero off-diagonal terms in $G$, implying 6 coupling terms between the three oscillators. In general, the higher the number of coupling terms, the more complicated the coupled array. Furthermore, the variable parameters $\epsilon$ and $\delta$ appear in several entries of $G$, meaning that they need to be varied over some range of parameters and at the same time be perfectly matched to each other in the corresponding coupling term. In particular, in [1] the exactly same value for $\epsilon$ (and for $\delta$) needs to be sent to the inputs of several matching analog multipliers. This matching problem becomes more problematic when the analog multipliers are replaced by physical resistors as it is common in several circuit implementations of coupled arrays.

3. A SIMPLER UNIVERSAL PROBE FOR SYNCHRONIZATION STABILITY

We propose other universal probes for synchronization stability which use the minimum number of coupling terms and minimize the occurrence of the same variable parameter in several coupling terms. The main idea is that other matrices $G$ exist which also allow Eq. (1) to be a universal probe. The only requirement on $G$ is that $G$ is a real-valued diagonalizable $3 \times 3$ zero sum matrix with two real parameters $\epsilon$ and $\delta$ such that the eigenvalues of $G$ are $\{0, \epsilon + i\delta, \epsilon - i\delta\}$. We will call this requirement $A$. It's clear that we only need to consider $\delta > 0$ as the case $\delta < 0$ is taken care of by the conjugate eigenvalues. This explains why all the boundaries between the stable and unstable region in the Master Stability Function plots in [1] are symmetric with respect to the real axis. In fact, more is true. All the specific matrices $G$ considered in this paper (Eqs. (3)-(4), Eqs. (6)-(9)) have the following property: $G$ corresponding to $\epsilon + i\delta$ is similar to $G$ corresponding to $\epsilon - i\delta$ via a permutation matrix\(^1\). In other words, the three oscillator universal probe for $\epsilon + i\delta$ is equivalent to the probe for $\epsilon - i\delta$ after relabeling the oscillators.

Consider the following $G$:

$$
G = \begin{pmatrix}
0 & 0 & 0 \\
\delta - \epsilon & \epsilon & -\delta \\
-\delta & \epsilon & \delta \\
\end{pmatrix}
$$

(4)

Note that this matrix $G$ satisfies requirement $A$ and has 4 nonzero off-diagonal terms. Furthermore, there is no coupling into the first oscillator. Its trajectories are those of an uncoupled oscillator. One possible implementation of the universal probe is to have the first oscillator replaced by a “memory” device which plays back the trajectory of an uncoupled oscillator. A schematic of this coupling scheme is shown in Fig. 2.

**Figure 2:** Schematic of coupling scheme for a simplified universal probe. The circles indicate the three oscillators and the labels on the edges are the weights of the corresponding coupling terms.

What is the minimum number of nonzero off-diagonal elements for matrices $G$ which satisfy requirement $A$? If a zero row sum matrix $G$ has two nonzero off-diagonal elements, they must be on different rows, as otherwise $G$ will have two zero eigenvalues. In this case $G$ takes on one of the following three forms:

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & a & -a \\
0 & b & -b \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
-\delta & \epsilon & -\delta \\
-\delta & \epsilon & \delta \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
-a & a & 0 \\
-b & 0 & b \\
\end{pmatrix}
$$

(5)

The first matrix has two zero eigenvalues, while the second and third matrices have as eigenvalues $\{0, a, b\}$ which are all real. Therefore $G$ must have at least nonzero 3 off-diagonal elements to satisfy requirement $A$.

If $\delta = 0$, i.e. the eigenvalue $\mu = \epsilon + i\delta$ is real, then a two oscillator probe with coupling matrix

$$
G = \begin{pmatrix}
0 & 0 \\
-\epsilon & \epsilon \\
\end{pmatrix}
$$

(6)

suffices as a universal probe. Let us assume that $\delta \neq 0$. In this case, consider the three oscillator probe (1) with coupling matrix:

$$
G = \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 1 \\
\epsilon^2 + \delta^2 & -(\epsilon + 1)^2 - \delta^2 & 2\epsilon + 1 \\
\end{pmatrix}
$$

(7)

Since $\delta \neq 0$, $G$ is diagonalizable and satisfies requirement $A$. $G$ also contains the minimal number of nonzero off-diagonal elements. By using the map $(\alpha, \beta) = (\epsilon^2 + \delta^2, -(\epsilon + 1)^2 - \delta^2) = V(\epsilon, \delta)$, $G$ can be written as

$$
G = \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 1 \\
\alpha & \beta & -\alpha - \beta \\
\end{pmatrix}
$$

(8)

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\(^1\)In fact, for Eqs. (7)-(9) the matrix $G$ corresponding to $\epsilon + i\delta$ is equal to $G$ corresponding to $\epsilon - i\delta$. 

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which minimizes the problem of having to match parameter values. Again, the first oscillator operates in the uncoupled mode. A schematic of this coupling scheme is shown in Fig. 3.

Figure 1: Universal probe consisting of 3 coupled Chua’s oscillators. The coupling matrix is given by Eq. (8) and the conductances of the coupling resistors are shown.

Figure 3: Schematic of coupling scheme for another simplified universal probe for the case when the eigenvalues are not real. In this scheme there are only three coupling elements between oscillators and a minimal number of coupling elements which need to be matched.

Written explicitly, the state equations of this universal probe are:

\[
\begin{align*}
\dot{x}_1 &= f(x_1) \\
\dot{x}_2 &= f(x_2) + H(x_3) - H(x_2) \\
\dot{x}_3 &= f(x_3) + \alpha (H(x_1) - H(x_3)) + \beta (H(x_2) - H(x_3))
\end{align*}
\]

To illustrate the simplicity of the coupling matrix (8), consider a coupled array of three Chua’s oscillators [5] where the coupling occurs between the state variable \(V_{C2} \) of each oscillator. The circuit diagram of the array is given in Fig. 1 with the coupling conductances shown. Note that since \(\alpha > 0, \beta < 0\), the two coupling resistors with conductances \(\alpha S\) and \((\beta - 1)S\) are passive and active respectively. The matching of the term \(-\alpha - \beta\) to the terms \(\alpha\) and \(\beta\) in Eq. (8) is satisfied automatically via Kirchoff’s laws.

Similarly, a universal probe for Chua’s oscillators using coupling matrix (4) can be implemented using a variable gyrator [6] as shown in Fig. 4. In some applications, the coupling between Chua’s oscillators occurs via passive resistors, resulting in \(G\) whose nonzero eigenvalues only have negative real parts, i.e., \(\epsilon < 0\). In this case the variable resistors in the universal probe in Fig. 4 are passive resistors.

Figure 5: Schematic of a coupling scheme for the case when the eigenvalues are not real nor purely imaginary. In this scheme there are 4 coupling elements in total.

Note that \(G\) in (7) has 5 nonzero entries. If \(\epsilon \neq 0\) and \(\delta \neq 0\), then

\[
G = \begin{pmatrix}
0 & 2\epsilon & -2\epsilon \\
0 & -\frac{\epsilon^2 + \delta^2}{2\epsilon} & \frac{\epsilon^2 + \delta^2}{2\epsilon} \\
-\frac{\epsilon^2 + \delta^2}{2\epsilon} & \frac{\epsilon^2 + \delta^2}{2\epsilon} & 0
\end{pmatrix}
\] (9)
Figure 4: Universal probe consisting of 3 coupled Chua's oscillators. The coupling matrix is given by Eq. (4) and the conductances of the coupling resistors and gyrator are shown.

satisfies requirement $A$ and has only 4 nonzero entries. This is schematically shown in Fig. 5. 4 is the minimum number of nonzero entries for a matrix $G$ satisfying requirement $A$ since $G$ must have at least one nonzero diagonal element (as the trace is nonzero) and 3 nonzero off-diagonal elements.

To illustrate these simpler universal probes, we simulated a three oscillator probe of piecewise-linear Rössler systems with $x$-coupling [1] using the coupling matrix in Eq. (9). The contour plot of the separation $S$ between the oscillators is shown in Fig. 6 which matches quite well with Figs. 6 and 8 in [1].

Figure 6: Contour plot of the separation $S$ between oscillators in a universal probe using Eq. (9).

4. CONCLUSIONS

We have presented three oscillator universal probes for synchronization stability in coupled arrays which are simpler than the one proposed in [1]. We presented probes which contain the minimal number of coupling terms and where matching of coupling terms is also minimized.

5. ACKNOWLEDGMENTS

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6. REFERENCES


