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MODELLING CONCEPTS ARISING FROM AN INVESTIGATION INTO A CHAOTIC SYSTEM

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Abstract—The Chua circuit, a simple third-order nonlinear dynamical system that exhibits chaotic behaviour, has highlighted shortcomings in presently used analysis and design techniques. We discuss the way in which chaotic dynamical behaviour in systems has affected many current ideas about the randomness of noise, the use of simulation, reducing problems to subproblems, the value of experiments, the use of nonlinear controllers, and the verification of system models.

1. INTRODUCTION

Having experimented and analysed Chua's circuit [1–5], a textbook example of a chaotic system, a number of points have arisen that challenge some of the existing engineering modelling and analysis concepts. This paper attempts to illustrate certain basic concepts of scientific theory that are emphasised by chaotic dynamical behaviour. In particular, physical modelling and simulation concepts are discussed and the importance of obtaining an appreciation of the use and limitations of modelling techniques is emphasised.

2. TRUTH IN SCIENCE

It is important to understand certain preliminary scientific concepts in order to obtain a better understanding of the modelling of physical systems and of chaotic dynamical systems in particular. To interpret and to carry out any basic scientific research, a sound background of the theory of knowledge is essential. One of the principles of the theory of knowledge that is important for physical modelling concerns the relationship between scientific theories and reality (true knowledge about the physical world). Pachner [6] reasons as follows.

Mathematical theorems and scientific theories are exact because they are logical constructions of the human mind. The choice of postulates or axioms, from which these theories are deduced, decides the region of validity of these theories and how well, if at all, they describe the physical world of experience. These scientific theories then become physical, chemical, or biological theories and are often called the "laws of nature." These theories cannot be extended beyond the boundaries of experience (regions of validity). *The question as to whether or not the laws of nature exist or whether these scientific theories describe absolute truth and knowledge of the physical world is meaningless.*

The word theory derives from the greek "theoria," which has the same root as "theatre," which means to view. Thus, theory is primarily a form of insight, of looking at the physical world, and not a form of true knowledge of how the world is in reality. Theories are valid within our boundaries of experience and become more and more unclear when extended beyond these boundaries of experience. Thus, our theories must be regarded primarily as ways of looking at the world as a whole, valid within our boundaries of experience, rather than as absolutely true knowledge of how the physical world is.

It is important to understand these ideas since unfortunately many hold a philosophy, denoted as "school philosophy," that asserts that there exists absolute true knowledge of the physical world, which science attempts to find. Since absolute true knowledge of the physical world

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implies the region of validity of science is infinite, this implies that there exists no such thing as a meaningless question for science. However, scientific theory is only a question of ensuring that the theory agrees with the human physical experience. The concept of absolute true knowledge of the physical world is meaningless in science and the theory of knowledge as stated previously.

This leads to the concept of truth as used in physical science. Physics seeks to understand the world on the basis of experimentation, measurement, and observation only. Anything outside the physical universe is thus not permitted to enter into any discussion. This implies that anything that cannot be measured or observed is considered as meaningless in physics.

A scientific theory or model is valid to the extent that it corresponds to experimental observation of physical phenomena. Physical phenomena are investigated in physics only in so far as it is possible to measure them and not with the impossible goal of describing their intimate essence. A physical quantity such as length, time, or mass is defined by prescribing the operations that are carried out in order to measure it. Length and time are meaningless quantities until we know how to measure them.

Consider Heisenberg's uncertainty principle in quantum mechanics, given by the relation

$$\Delta p \Delta s \geq h \tag{1}$$

where h is Planck's constant, Δp the uncertainty in a particle's momentum, and Δs the uncertainty in its position. Thus, if position is measured exactly, that is $\Delta s \to 0$, then $\Delta p \to \infty$, implying that it is impossible to measure or say anything precise about the particle's momentum. When this is the case, we say that the particle's precise momentum is a meaningless concept, since it cannot be measured and hence defined. We do not say that momentum exists and is a meaningful concept that we cannot measure with existing equipment, but rather the very concept is meaningless and hence cannot be spoken about scientifically. The consequence of using meaningless concepts is that it leads to absurd results.

Physics does not measure quantities in order to reveal their intrinsic essence, but with the more modest aim of comparing the results of these measurements in order to discover the mathematical relations existing between them. A typical law of physics consists of a mathematical relation found to exist between the measurements of the various quantities that take part in the phenomenon. The specification of the limits of validity (boundary of experience) of the law and the precision ε with which it has been verified represents an integral and essential part of the physical method. If the measurements are made with a better precision ϵ' where $\epsilon > \epsilon'$, the domain of validity has changed and there is no certainty that the physical law will still be valid.

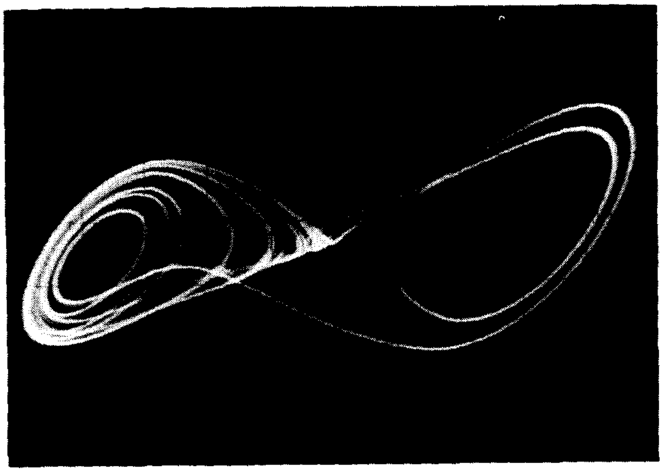
With these concepts in mind, we can now proceed to investigate some concepts arising from chaotic dynamical systems.

3. WHAT IS CHAOS?

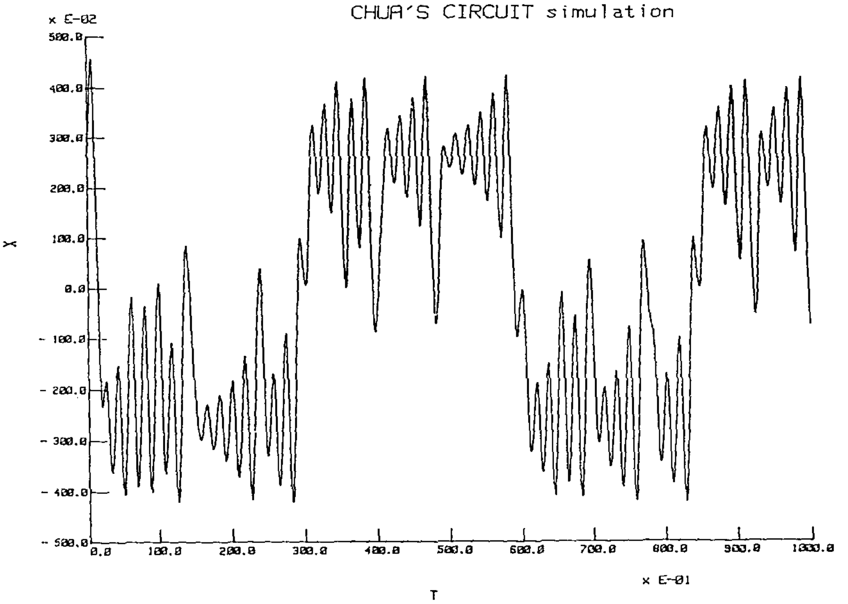
There is still no universally accepted definition for chaos [7]. The definition given here is more in terms of the descriptive properties of chaotic nonlinear systems.

First, it should be noted that chaos is a form of steady-state behaviour with some similarities to limit cycles. There are, however, distinct differences. Chaos is a form of osscillation, where the oscillation is bounded but aperiodic, as illustrated in Figure 1. It should be noted that apparent aperiodicity in a time waveform is not sufficient to define a system as chaotic. One could be dealing with a periodic oscillation with a long period. True aperiodicity is manifested in a continuous frequency spectrum. This is an essential condition for an oscillation to be chaotic.

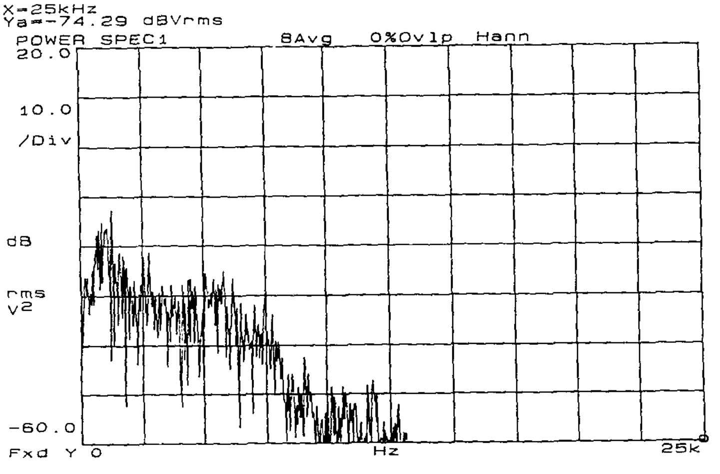
Extreme sensitivity to initial conditions is another major feature of chaotic systems. Starting off with two initial conditions arbitrarily close to each other in a chaotic system, the trajectories will diverge at a rate characteristic of the system, until for all practical purposes they become completely uncorrelated. This is clearly illustrated in Figure 2. In theory, the trajectory of the physical system exhibiting chaotic behaviour can only be quantitatively determined if the system's initial conditions are known with *infinite accuracy*. However, in practice, the physical system's initial conditions can only be measured or defined with finite accuracy. The result is that no matter to what finite precision the initial conditions of a physical system areknown, the long-term dynamical behaviour can *never* be quantitatively predicted. This explains why chaotic systems are sometimes referred to as deterministic systems exhibiting random behaviour.



(a) A chaotic phase plane observed experimentally.



(b) Time waveform of a chaotic state variable.



(c) Frequency spectrum of a chaotic state.

Fig. 1. Experimental results of Chua's circuit exhibiting chaotic behavior.

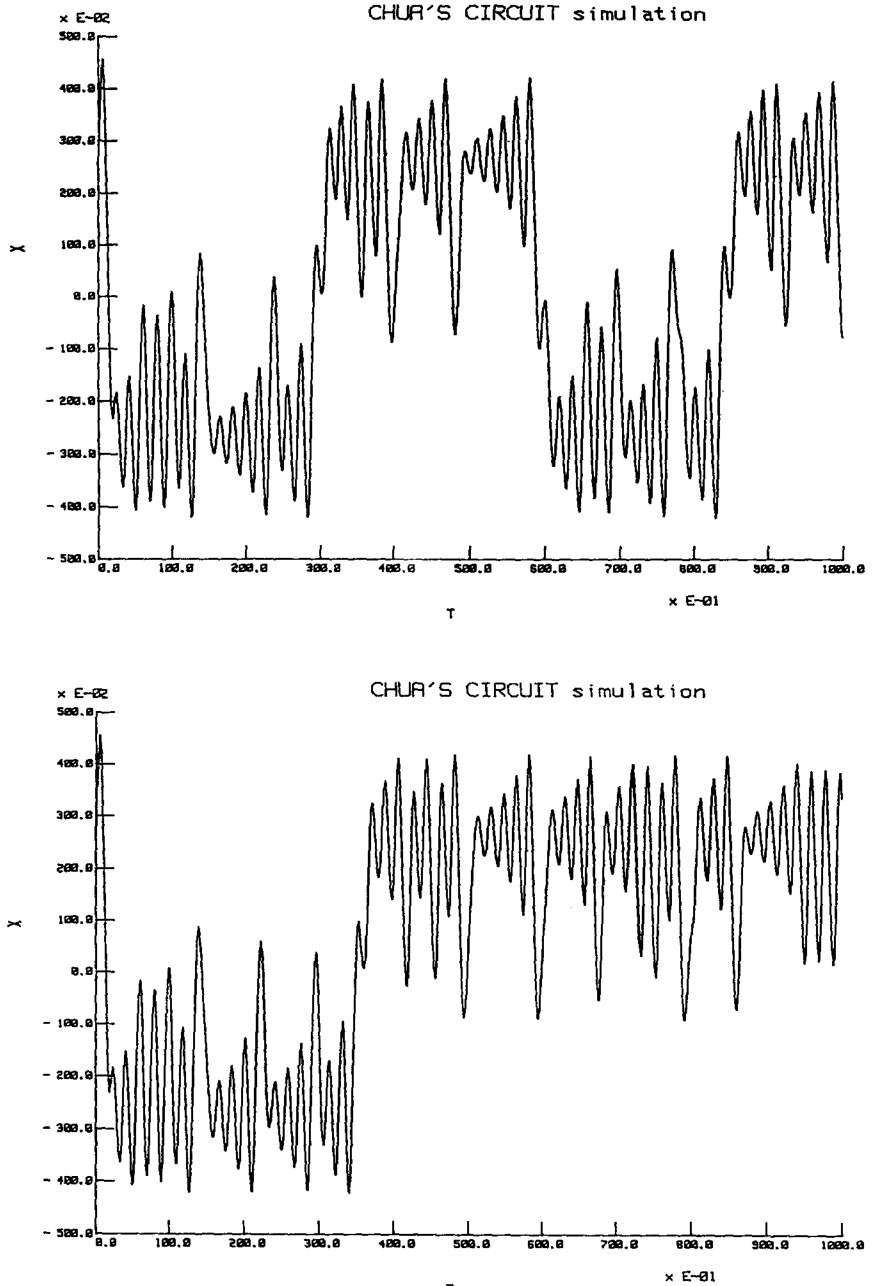


Fig. 2. Sensitivity dependence on initial conditions. The initial condition was changed by 0.01%. Initially, the waveforms are identical, becoming totally uncorrelated after about 20 seconds.

If a physical system fulfils both the criteria of extreme sensitivity to initial conditions as well as aperiodic oscillation, then the system can be regarded as chaotic. What gives rise to chaotic behaviour? This question still cannot be answered in general, but some guidelines can be given. Chaos always seems to arise out of complex interactions between different regions of dynamic nonlinear behaviour. In particular, it can occur in strongly nonlinear systems with feedback, as illustrated in Section 5.

4. INTRODUCING THE CHUA CIRCUIT

If Chaotic behaviour were a pathologically concocted oddity, its practical significance would be very limited. However, it is a form of steady-state behaviour possible in every nonlinear continuous system, provided that its order is higher than two if it is a forced circuit or three if the circuit is autonomous and certain conditions are met.

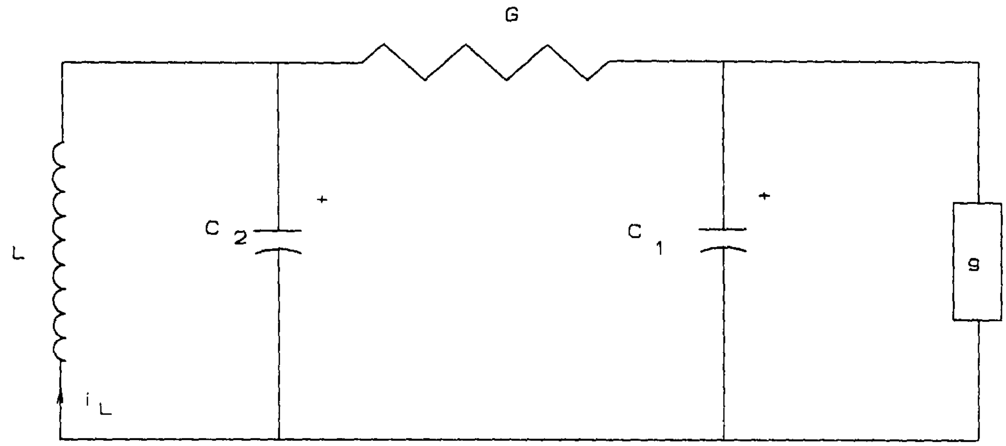


Fig. 3. Chua's circuit.

A circuit which exhibits chaotic dynamical behaviour [1–3] is the Chua circuit, as shown in Figure 3. This circuit is a good textbook example of a chaotic system, due to its being the only known system whose chaotic behaviour has been verified by rigorous mathematics [2], simulation, and experiment [1,3–5,8,9]. Chua's circuit was tested and its chaotic behaviour experimentally verified [1], as shown in Figure 1. As shown in Figure 3, Chua's circuit is very simple. Element g represents a nonlinear resistor with the characteristic shown in Figure 4. Choosing the state variables of the system as the voltages across the capacitors and the current through the inductor, the state equations of the system may be written in the normalised form [1]

$$\begin{aligned}\frac{dx}{d\tau} &= \alpha(y - x - h(x))\\ \frac{dy}{d\tau} &= x - y + z\\ \frac{dz}{d\tau} &= -\beta y \end{aligned} \tag{2}$$

where $\alpha = C_2/C_1$, $\beta = C_2/LG^2$, $\tau = tG/C_2$, $h(x) = i_g/G$, $x - V_{C_1}$, $y = V_{C_2}$, $z = i_L/G$.

It should be noted that the chaotic behaviour of the system (equation 2) is a strong function of the nonlinear resistor characteristic (Figure 4) as discussed in Section 5. Analytically, it can be shown that the slopes of the nonlinear resistor characteristic can only vary within very specific limits for chaotic behaviour to occur. The criterion for chaotic behaviour to occur was seen to be that the line $h(x) = -x$ intersects in sections B and D of the characteristic, as shown in Figure 4. Note that there are many other and even simpler systems in which chaotic behaviour has been reported [8,10], such as a circuit with a driven diode and inductance [8]. The fact that many other examples of chaotic systems can be found clearly illustrates that the Chua circuit is not a rare oddity.

5. THE IMPACT OF CHAOS ON MODELLING PHILOSOPHIES

This section outlines how the discovery of chaos has forced a revision of existing modelling concepts and discusses the impact it makes on the analysis and design approaches currently used by engineers. Based on the concept of truth in science as introduced in Section 2, it is quite easy to accept the results arising from chaotic systems.

5.1. Quantitative Prediction using Simulation

Even in very simple systems in regions of chaotic behaviour, knowledge of the system's initial conditions does not allow the quantitative prediction of all future system behaviour from the

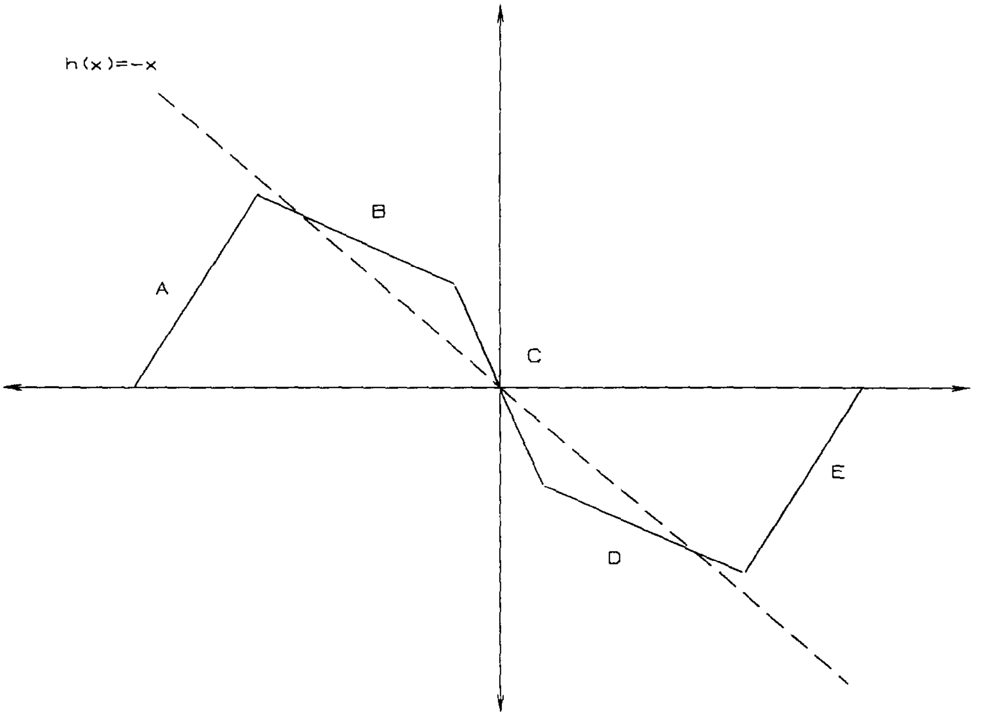


Fig. 4. The nonlinear resistor characteristic in Chua's circuit where chaos can occur.

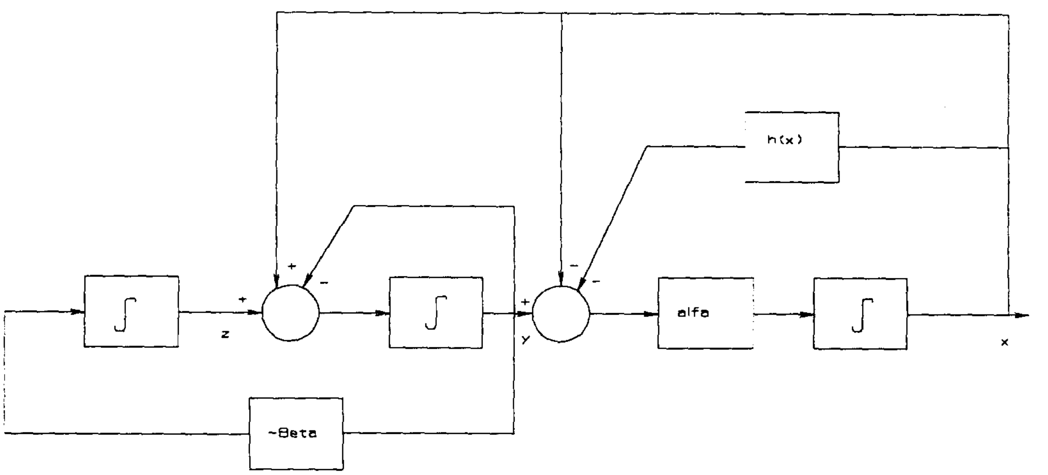


Fig. 5. Block diagram of Chua's circuit.

deterministic system equations. Recall from Section 3 that this would require infinite accuracy in the initial conditions, which is unrealistic practically. Consider the example of predicting the weather, a far more complicated system than the Chua circuit. It is possible, knowing the present weather (initial conditions), to predict with some certainty the conditions for the next few hours, but impossible to make accurate long-term predictions. *Even if we had an infinitely powerful computer and an accurate meteorological simulation model of the weather, it will never be possible to predict the weather without having infinitely accurate measurements of the initial conditions.*

This sobering example has a number of serious consequences for the use of simulation in engineering analysis and design. In many fields of engineering it has become common practice to use simulation as a tool for verifying designs. Simulation is also used to establish what the effect on the dynamical behaviour of the physical plant is when system parameters are varied.

This technique relies strongly on the fact that physical models have quantitative predictive value for all future time. In systems in which chaotic behaviour occurs, simulation is dangerous, since it cannot answer the "what happens if" question unless the system initial conditions are known with infinite accuracy, a physically meaningless concept.

Gleick [11] argues that it was the inability of deterministic mathematical equations to do quantitative predictions that made Lorenz and other chaos pioneers resist what they were seeing. However, are these results all that surprising? Recall the definition of scientific truth as given in Section 2. Science can develop models based on experimental results. Its truth does not purport to be the absolute truth about the physical laws governing nature and therefore quantitative prediction is not guaranteed. Using the deterministic model equations in a chaotic system to quantitatively predict the system's dynamical behaviour requires infinite accuracy of the system's initial conditions, which is a meaningless concept in physics (Section 2). *Hence, the concept of using a deterministic model to predict the quantitative dynamical behaviour of a physical system exhibiting chaotic behaviour is meaningless and leads to absurd results.*

Thus, having a precise system model does not necessarily mean that simulation can be used to establish the correct quantitative dynamical behaviour of a physical system. Numerically solving the model equation of a physical system in a chaotic region is not the best approach for analysing the system's behaviour. In systems that are not in chaotic regions of operation, simulation can be used to predict a system's quantitative dynamical behaviour, which is the reason why simulation has become so popular. Currently, there are no criteria for establishing when the nonlinearity becomes "strong" enough for simulation to become meaningless for quantitative prediction of the dynamical system behaviour of the physical process.

5.2. Random Noise

Chaos is manifested as something close to random noise. This is the result of the aperiodicity of the oscillation. As can be seen from Figure 1, it could be quite easy to wrongly identify chaos as bandlimited white noise and thus incorrectly analyse a system's chaotic dynamical behaviour as random noise. At present, many engineers are unaware of the possibility that what is called random noise in a system may in fact be a manifestation of the system's dynamical behaviour.

The fact that it is possible for a system to oscillate continuously in an aperiodic manner means the physical system has nonstationary statistical behaviour. This makes stationary statistical analysis of an apparently random process (bandlimited white noise) impossible. Recall that for an aperiodic waveform, no average can be determined. This means that no statistical distribution curve can be fitted to chaotic behaviour. Thus ergodicity, an assumption used in the analysis of most engineering systems with noise, does not hold.

5.3. Reductional Philosophy and Linearisation in Modelling

Chaos has also made a major impact on physical system modelling. The popular approach to analysis and design of most systems today is to use reductional principles. Here a problem is broken up into its constituent components or modules. The reductional philosophy assumes that understanding dynamical behaviour of the basic building block will enable an understanding of complex system dynamical behaviour, where the complex system is constructed of a combination of different basic building blocks. This manner of thinking has led physicists to reduce problems to subatomic constituent elements in order to understand the physical world on a macroscopic scale, the idea being that all physical elements are constructed of some basic indivisible element called the basic building block of the universe. Nonlinear systems exhibiting chaotic dynamical behaviour have shown that this approach to analysing systems is useless. In order to successfully model and understand the behaviour of nonlinear dynamical systems exhibiting chaotic behaviour, a global approach to the analysis of the physical system is required. This may be illustrated as follows. Figures 6 and 7 contain plots of the nonlinear resistor characteristic and eigenvalues as a function of β of the different equilibrium points of the Chua circuit, respectively.

The equilibrium points of the Chua circuit are the values of the states (x, y, z) such that

$$\begin{aligned} \alpha(y - x - h(x)) &= 0 \\ x - y + z &= 0 \\ -\beta y &= 0 \end{aligned} \tag{3}$$

Equation 2 describes the dynamical evolution of the states (x, y, z) with time and equation 3 the steady-state equilibrium values of the states of the Chua circuit (Figure 3). Chua's circuit has either one or three equilibrium points, one of which is always at the centre while the other two, if they exist, are located in an odd symmetrical manner on the nonlinear resistor characteristic, as shown in Figure 6. Since Chua's circuit, described by equation 2, is third-order, each equilibrium point given by equation 3 has three eigenvalues, one real and a pair of complex conjugate eigenvalues.

Figure 6a shows the Chua circuit with three equilibrium points, two outer ones in Regions A and E of the nonlinear resistor characteristic and one at the centre in region C. Two corresponding eigenvalue plots are each provided for the centre equilibrium point $(0; 0; 0)$ and for one of the outer equilibrium points $(6,53; 0; -6,53)$, as shown in Figure 7a. These eigenvalue plots show that the centre equilibrium point is unstable, while the outer equilibrium points have their real parts in the left-hand complex plane, implying that they are stable. The dynamical behaviour obtained from simulations under these conditions was as expected from analysing the constituent portions. The system exhibited stable exponentially oscillatory behaviour being attracted to either one of the outer equilibrium points (regions A or E) dependent on the initial conditions.

In Figure 6b, the outer equilibrium points have moved to regions B and D of the nonlinear resistor characteristic. Two corresponding eigenvalue plots are each provided for the centre equilibrium point (0;0;0) and for one of the outer equilibrium points $(2,69; 0; -2,69)$, as shown in Figure 7b. This is the case where double scrollings as well as chaotic dynamical behaviour were observed from simulation and experimental results, as shown in Figure 1. The eigenvalues of the centre equilibrium point again classify it as unstable. At the outer equilibrium points (regions B and D), a Hopf bifurcation, also called a limit cycle, is obtained. Analysing the system as a linear system implies that the system is unstable at each equilibrium point. Nonlinear analysis implies that a limit cycle exists around each of the outer equilibrium points (regions B and D). The interaction of the outer equilibrium points produce a double scroll trajectory in the phase plane (Figure 1a), which cannot be predicted unless the complete system is analysed. It is impossible to predict the full picture of the dynamical behaviour of a system by analysing its constituents individually and based on this try and surmise how the complete system will respond.

Figure 6c represents the case where only a single equilibrium point exists at the centre in region C. The corresponding eigenvalue plots are provided for the centre equilibrium point (0;0;0), as shown in Figure 7c. The eigenvalues again reveal a limit cycle, which in this case is unstable. Here linear analysis gives the correct information since the system is unstable, but would not have if the limit cycle was stable.

From the above discussion it becomes clear that analysing constituent elements in isolation is not sufficient to reveal the dynamical behaviour of a system. Further, using the common engineering approach of linearising the constituent elements results in an approach that is totally inadequate for the physical modelling of nonlinear systems if used blindly. The problem is to establish criteria for when linear analysis gives valid results (Figures 6a and c) and when it does not (Figure 6b). This question is perhaps one of the most important questions that future systems research needs to answer.

5.4. Controller Design

In the field of controller design, there is also a lesson to be learned from chaotic systems. In nature it is observed that feedback is frequently performed by nonlinear controllers. This fact, and the requirement for improved system performance, has led to the suggestion that nonlinear feedback controllers should be used in control design. Although this is indeed true, a word of warning is necessary. Consider the block diagram of the Chua circuit as shown in Figure 5.

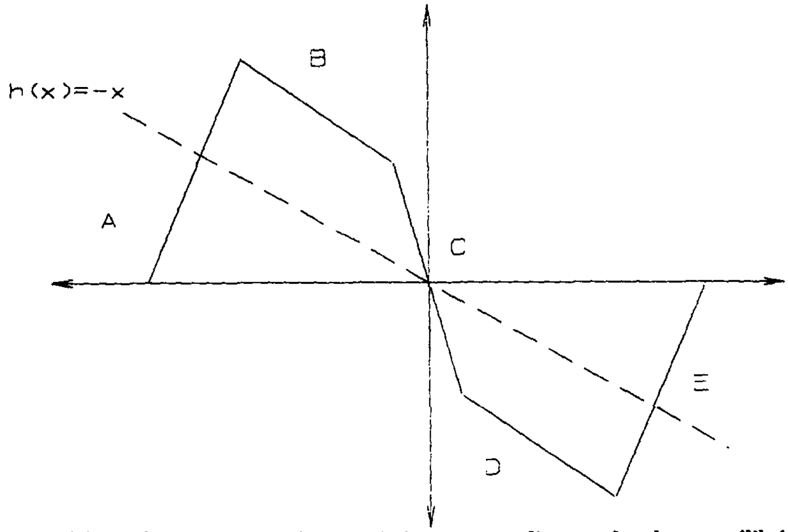


Fig. 6.(a) Nonlinear resistor characteristic corresponding to the three equilibrium points at (6, 53; 0; −6, 53), (−6, 53; 0; 6, 53), and (0;0;0).

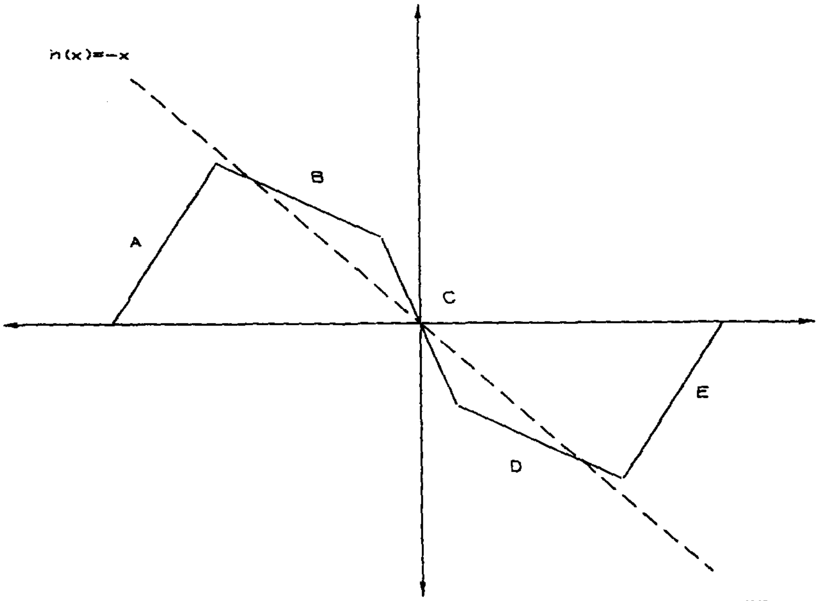


Fig. 6(b) Nonlinear resistor characteristic corresponding to the three equilibrium points at (2, 69; 0; −2, 69), (−2, 69; 0; 2, 69), and (0;0;0).

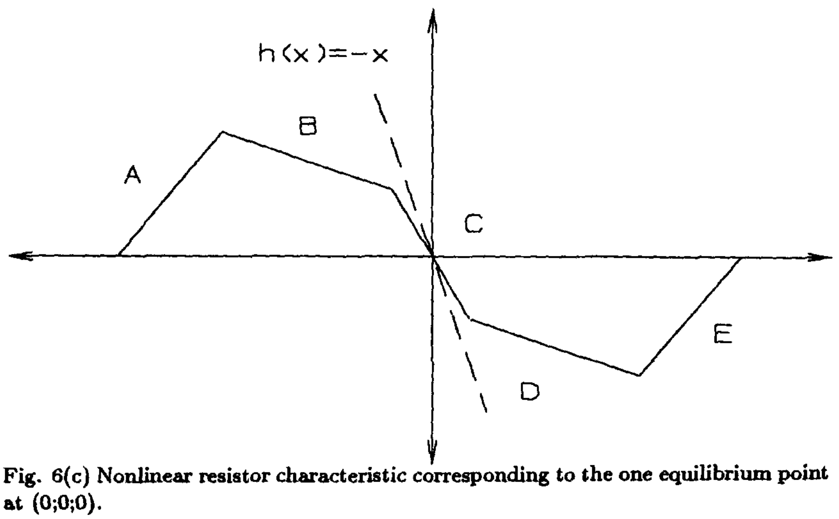
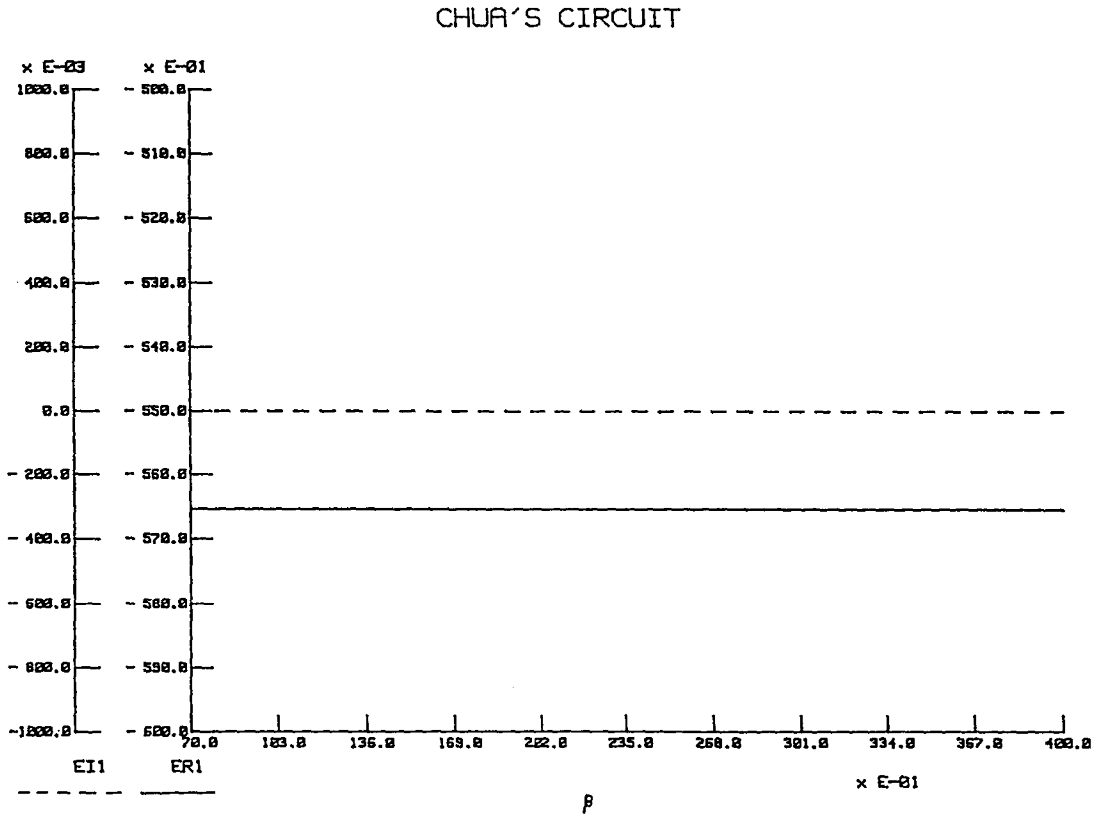


Fig. 6(c) Nonlinear resistor characteristic corresponding to the one equilibrium point at (0;0;0).



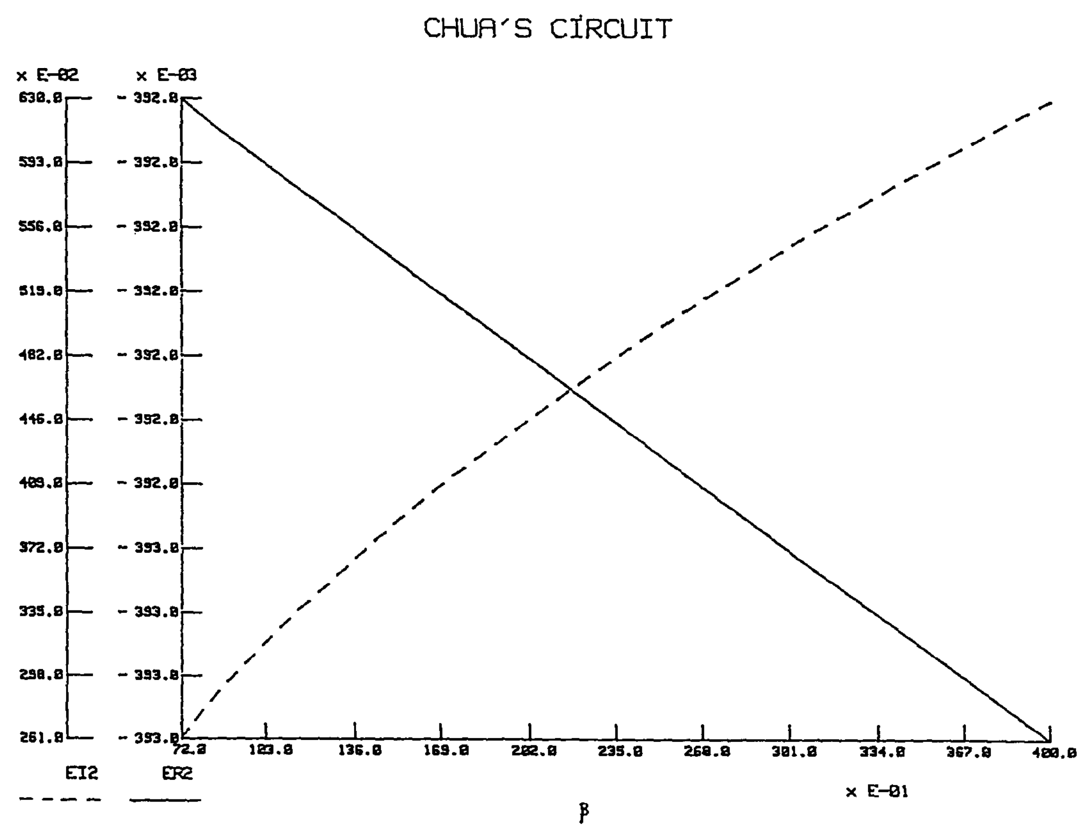
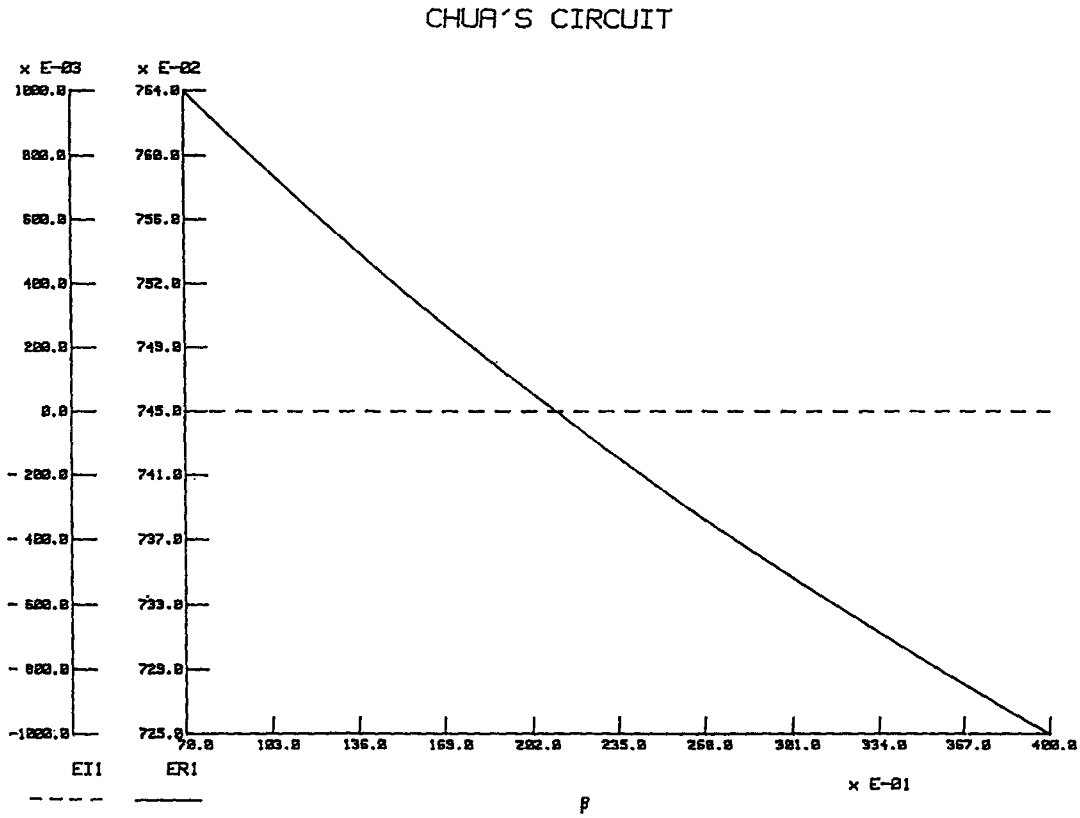


Fig. 7a. Eigenvalue plots as a function of β for the equilibrium point at $(6,53;0;-6,53)$ for a $h(x)$ as per Fig. 6a.



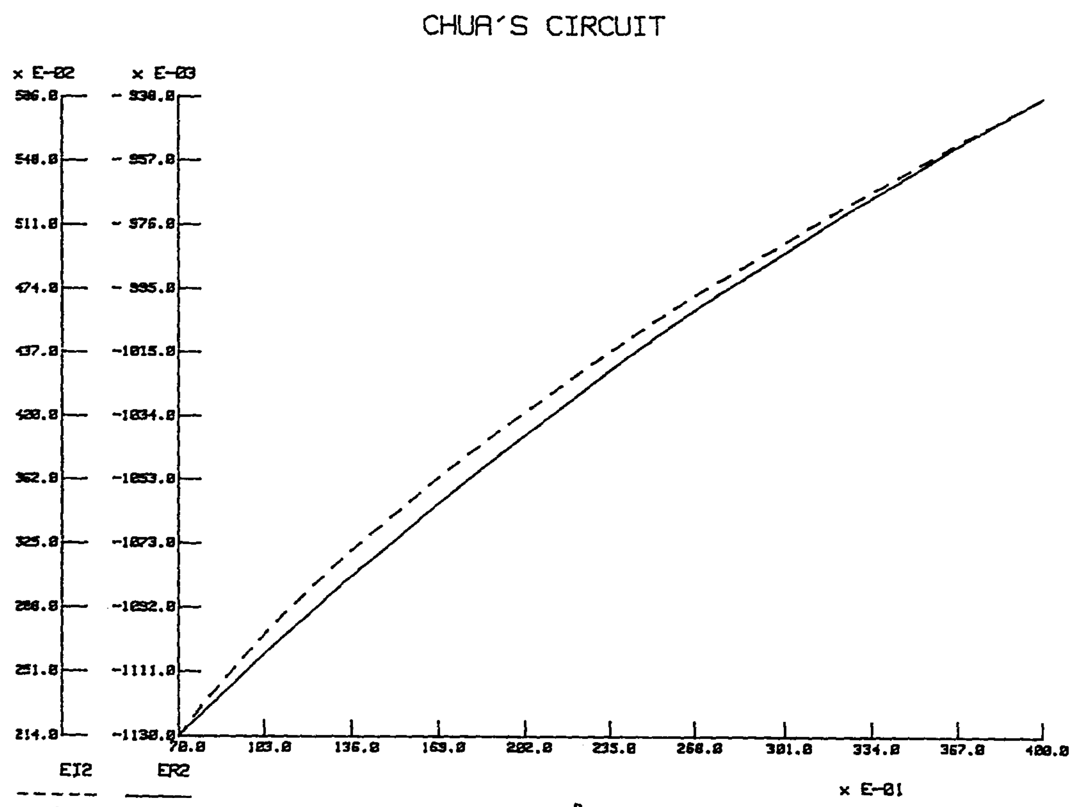
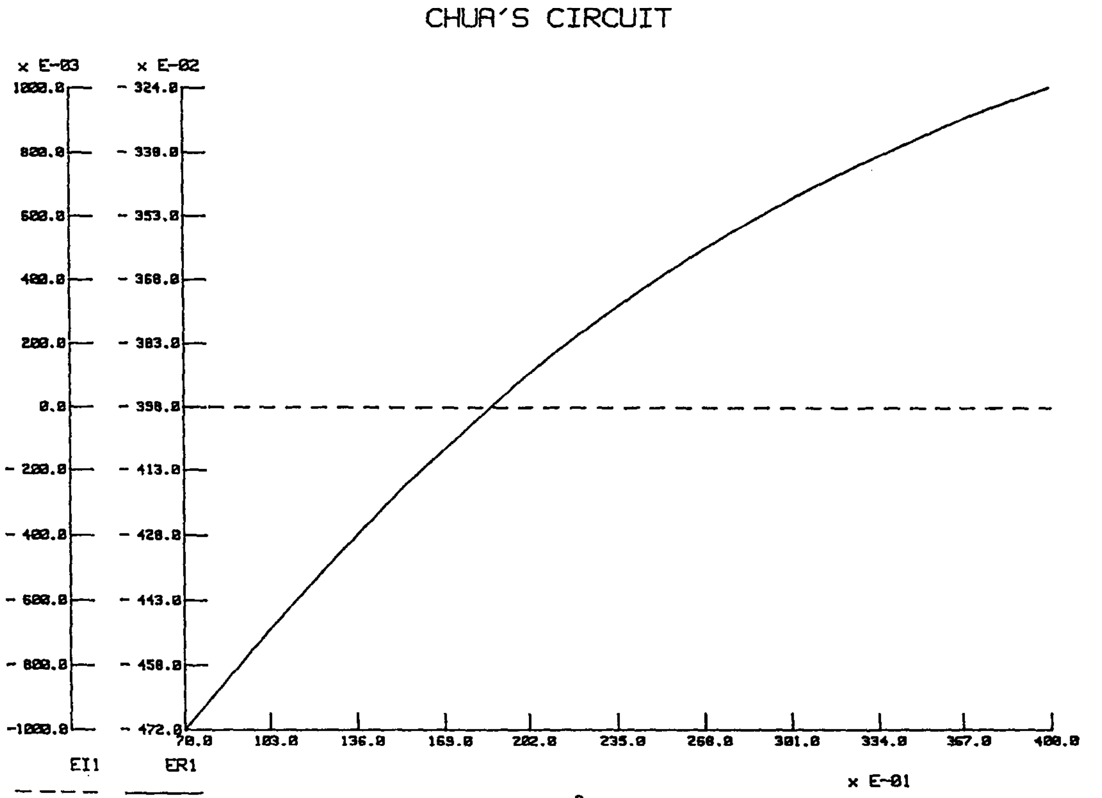


Fig. 7a. (*continued*) Eigenvalue plots as a function of β for the equilibrium point at $(0;0;0)$ for a $h(x)$ as per Fig. 6a.



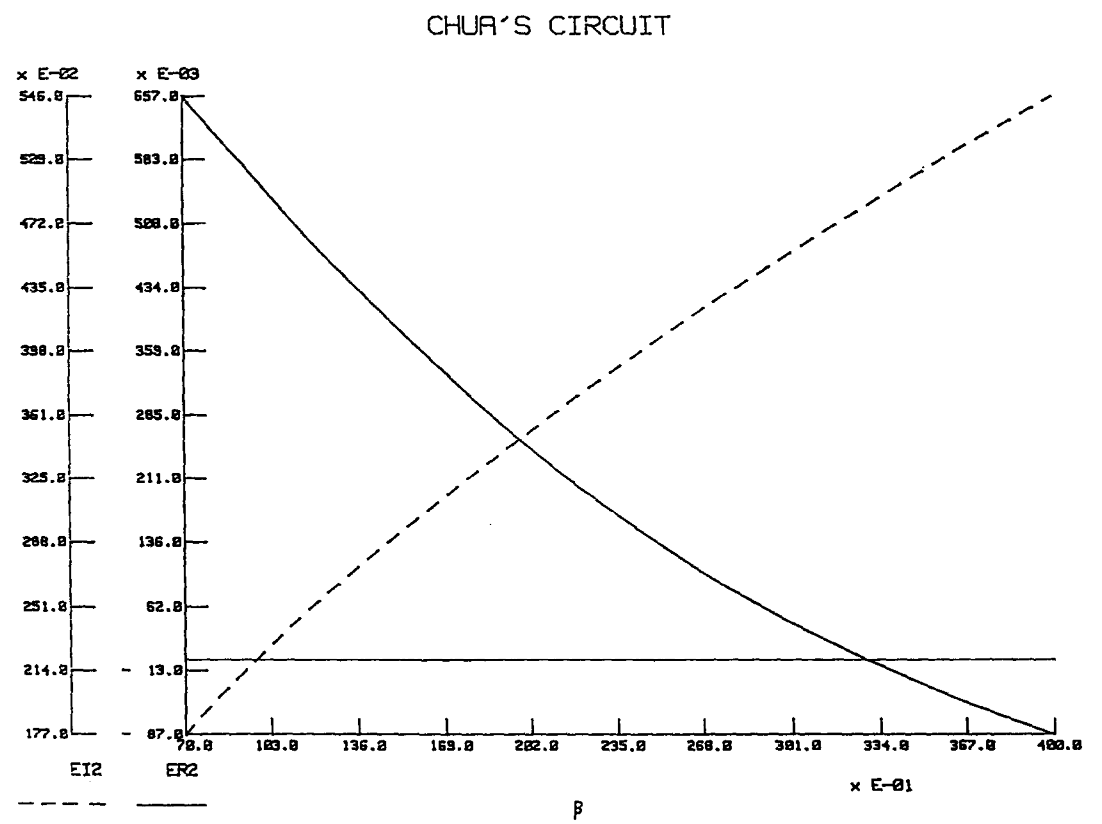
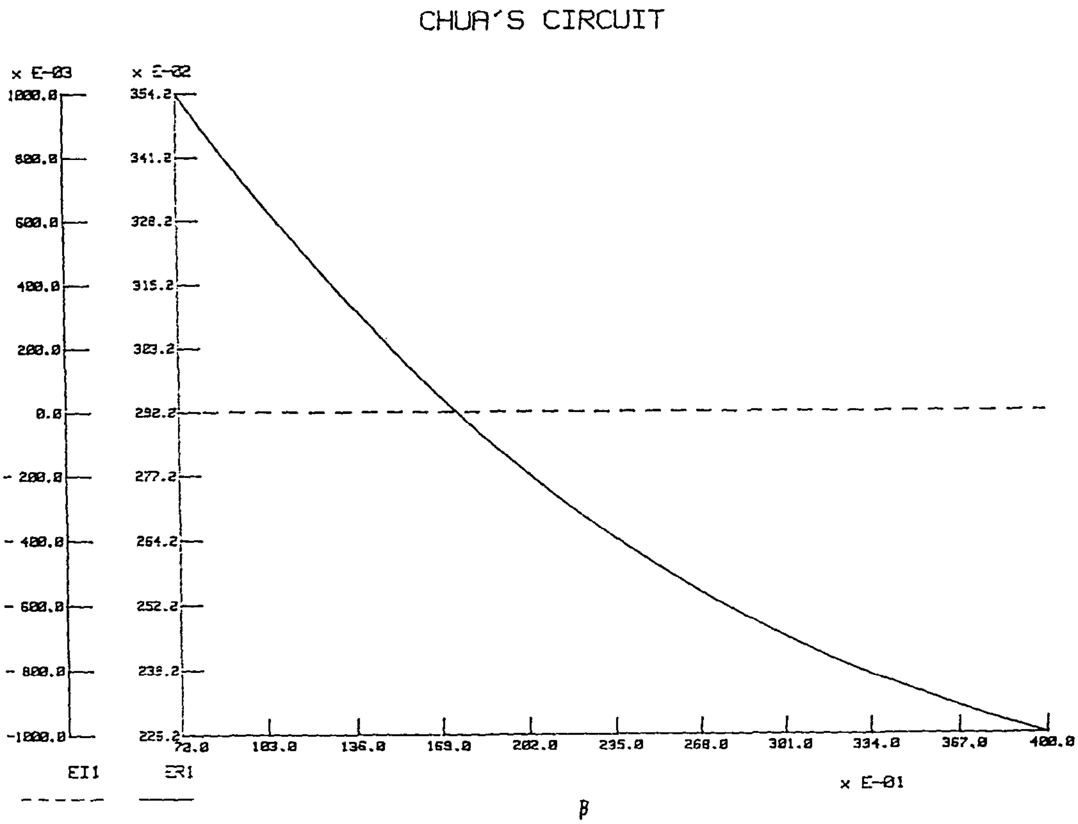


Fig. 7b. Eigenvalue plots as a function of β for the equilibrium point at $(2,69;0;-2,69)$ for a $h(x)$ as per Fig. 6b.



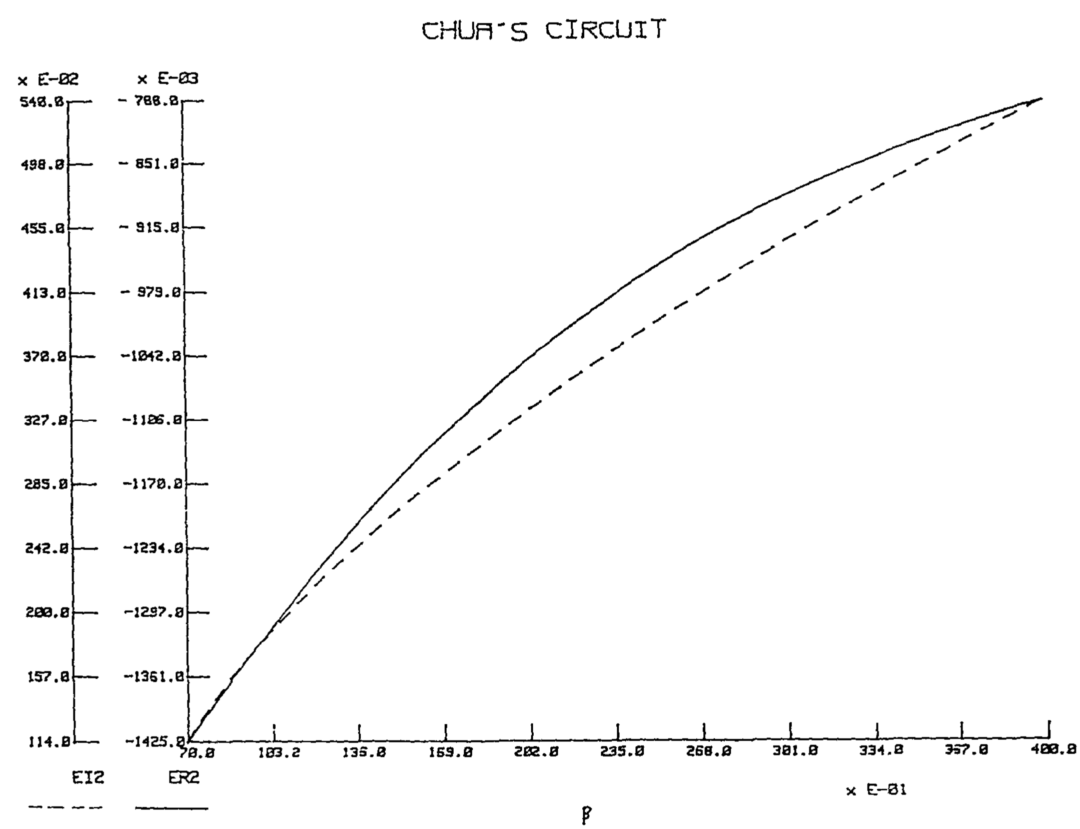
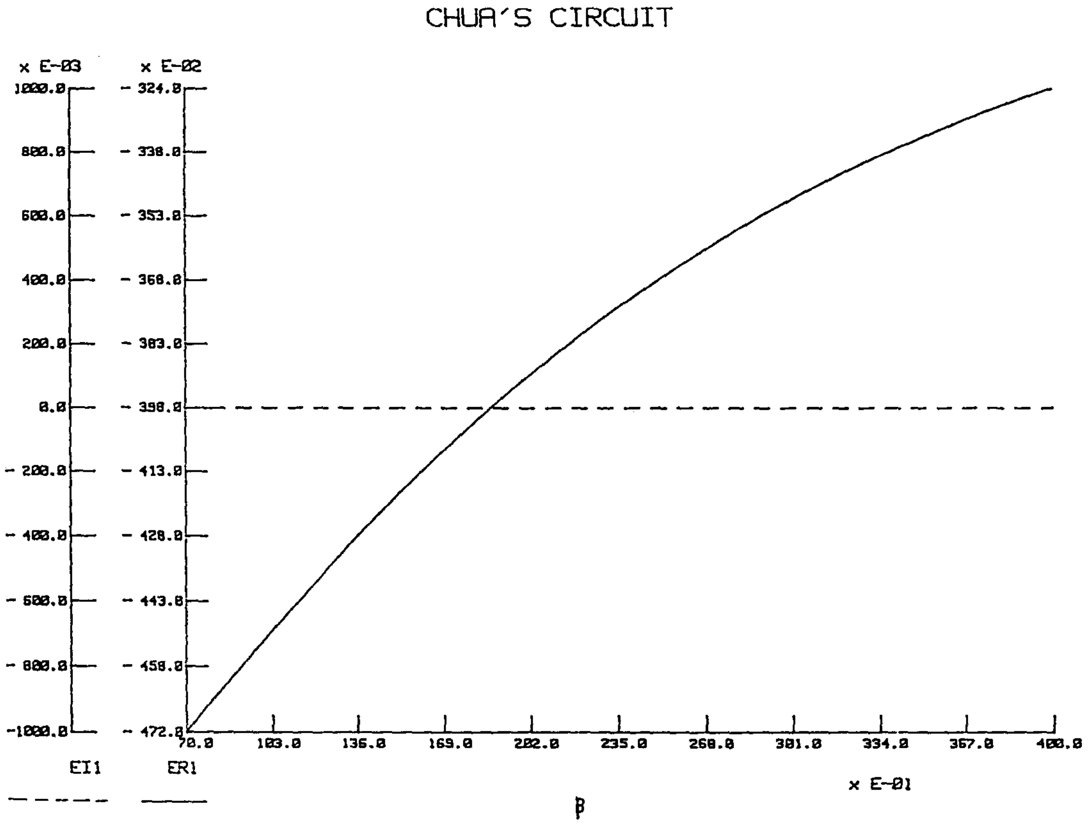


Fig. 7b. (*continued*) Eigenvalue plots as a function of β for the equilibrium point at $(0;0;0)$ for a $h(x)$ as per Fig. 6b.



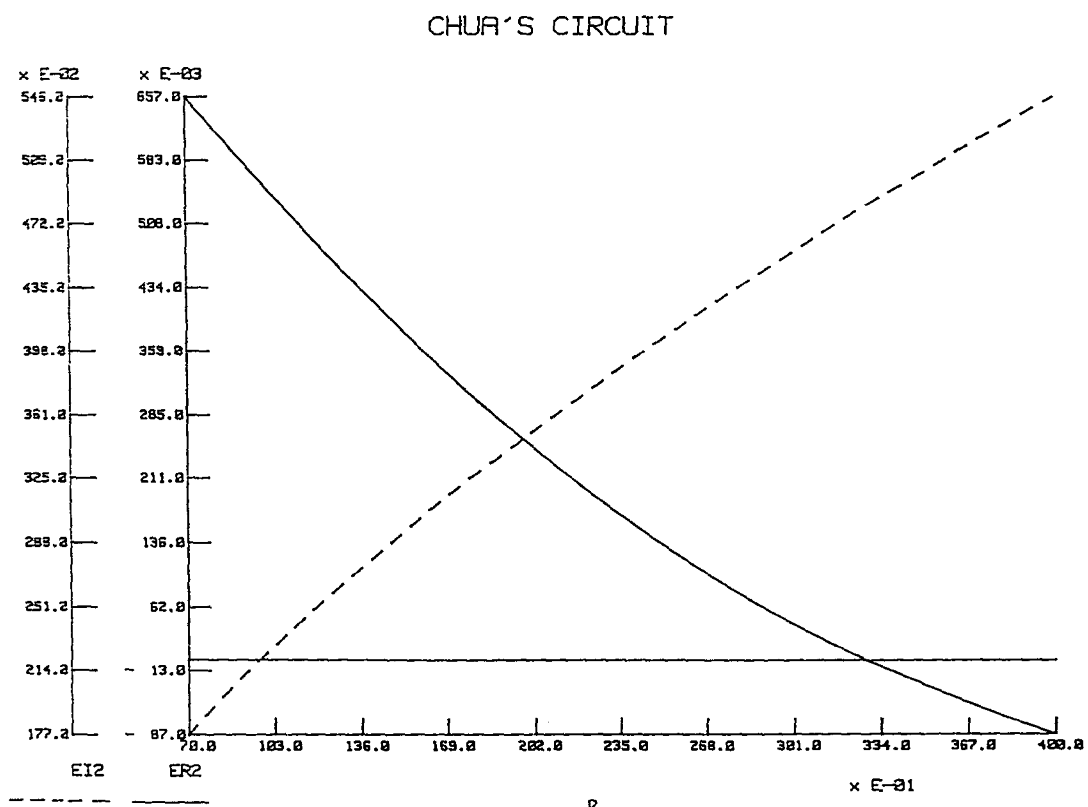


Fig. 7c. Eigenvalue plots as a function of β for the equilibrium point at $(0;0;0)$ for a $h(x)$ as per Fig. 6c.

Clearly, the Chua circuit has nonlinear feedback in the model. The nonlinear elements are responsible for all the complex dynamics seen in the circuit behaviour. When purposefully introducing nonlinearities to a system by nonlinear controller design, the designer should be aware of the potential complexities introduced. A far stronger motivation for nonlinear controller design is necessary than just a possible improvement in performance before the risks associated with this approach become warranted.

An important aspect that chaotic systems have brought to light is the inadequacy of existing nonlinear analysis, in particular controller design techniques for nonlinear systems. It is presently difficult to predict whether chaotic dynamical behaviour can exist in a system. Having observed chaotic behaviour, there are a number of techniques for confirming that what we observed was indeed chaos, but even these are seldomly applicable to any general system and are computationally complex. The seemingly general occurrence of chaos may point to the fact that in a proper understanding of chaotic dynamical behaviour, the stepping stone toward a nonlinear systems theory as widely applicable as linear systems theory today may be found.

5.5. Model Verification with the Physical Process

A further area in which chaos has made an impact is the viewing of experimental data. In the past it was believed that once experimental data appeared erratic, it was corrupted by noise. Chaos has shown that this need not be the case. Apart from this, experimentation can be dangerous because the data can be read to give the required results. For example, when a linear relationship is required, great effort is made to regress data points to fit a straight line. All points slightly differing from the expected relationship are attributed to experimental error or random effects such as noise. It is primarily due to these reasons that chaos was only recognised as recently as 15 years ago.

Simplified models of a physical process which are valid in a certain region of operation are commonly used in engineering for analysis and design. This corresponds to deliberately limiting the "domain of experience" for which the model/theory is valid. The model must then be verified with the physical model to be a good approximation in the region of operation. This is in line with the scientific method as outlined in Section 2. In chaotic dynamical systems, however, quantitatively verifying the model is extremely difficult due to the system's sensitivity to the initial conditions.

The question then asked is how a system model is conclusively verified. A system model is verified only if mathematical proof, computer simulation, and experimental data simultaneously agree completely about the dynamical behaviour of the system. Due to experimental error primarily occurring due to ignored dynamic phenomena, this is virtually impossible to achieve for physical systems exhibiting chaotic dynamical behaviour. The difficulty of verifying a chaotic dynamical system model is outlined clearly by the fact that Chua's circuit is the only known system whose chaotic dynamical behaviour has been proved by rigorous mathematics [2]. Even here it is not possible to get simulation and experiment to agree completely because the infinite accuracy with which the initial conditions need to be known cannot be attained [1]. Even in systems as simple as the Chua circuit we thus have difficulty in obtaining verification, let alone "proof," that the model is correct. In most cases, a qualitative verification is all that can be obtained. The adjective qualitative is used to denote the inherent local features and properties of the behaviour of dynamical systems about an operating point. A local property is valid in a neighbourhood of a point, the size of which is not specified. A model would thus be regarded as realistic if the mathematics, simulation, and experiment gave *qualitatively* similar results. This is the manner in which Chua's circuit was verified to be chaotic [2,3,9].

Why are the three criteria outlined above regarded as necessary for satisfactorily assuming that a system model is indeed correct? Laboratory experiments can be strongly affected by external unmodelled phenomena. Accumulation of rounding errors, as well as initial conditions, may strongly affect computer simulation. It is not possible to refute mathematical proof as such, but it is quite possible for the model to fail to represent the actual physical system. Hence, qualitative agreement is required to overcome the uncertainty in any one method. It is precisely the difficulty in obtaining a correct model that in most chaotic systems provides so much difficulty.

6. CONCLUSIONS

We live in a world where higher performance is increasingly being demanded from all physical systems. This means that many physical systems are now operating in regions where linearity is not a good approximation for describing dynamical behaviour. All systems are essentially nonlinear, some of them having regions where linear behaviour is a good approximation. Chaotic dynamical behaviour in simple and complex systems has illustrated the importance of obtaining a good understanding of the dynamical behaviour of nonlinear systems for engineering applications.

Chaotic systems have resulted in a revision of some physical modelling and analysis concepts. In particular, the idea of using simulation for quantitative prediction of the dynamical behaviour of systems exhibiting chaotic behaviour is a meaningless concept.

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