Chua's Circuit: Rigorous Results and Future Problems

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Invited Paper

Abstract—**This paper provides a mathematician's perspective on Chua's circuit as a paradigm for chaos. It explains why Chua's circuit is of interest not only to radio physicists but to nonlinear scientists from other disciplines as well. It points out why the double-scroll Chua's attractor, which has been proved rigorously to be chaotic in [5] in the sense of the Shil'nikov theorem, is fundamentally different and mathematically much more complicated than that of the Lorenz attractor.**

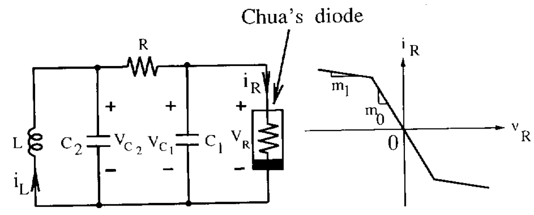

Fig. 1. (a) Chua's circuit; (b) voltage versus current characteristic of the Chua's diode may be *any* nonlinear function, e.g., a polynomial, a "cubic" $f(x) = c_0x + c_1x^3$, a "sigmoid" $f(x) = \frac{exp(\gamma x)-1.0}{exp(\gamma x)+1.0}$, etc.[2]. Here, we show the most commonly chosen piecewise-linear characteristic.

I. Introduction

One of the most remarkable achievements of science in the 20th century is the discovery of dynamical chaos. Using this paradigm, many of the problems in modern science and engineering that can be modeled via the language of nonlinear dynamics have attained an adequate mathematical description. However, the explanation of a number of phenomena of dynamical chaos has required the creation of new mathematical techniques. The reason for this is that the classical theory of nonlinear oscillations developed by Van der Pol *et al.* was based on Poincaré's theory of periodic orbits and Lyapunov's stability theory, i.e., on methods for studying mainly *quasilinear* systems.

Problems associated with systems involving high energies, powers, velocities, etc. must be modeled by multidimensional and *strongly nonlinear* differential equations (ordinary, partial, etc.). The study of such systems has generated numerous new concepts and terminology: hyperbolic sets, symbolic dynamics, homo- and heteroclinic orbits, global bifurcations, entropy (topological and metric), Lyapunov exponents, fractal dimension, etc. We note the possibility of describing dynamical chaos via statistical tools as well, e.g., correlation function, power spectrum, etc. They are widely used in numerical simulations and in experiments.

Here, an important role should be noted on which concrete phenomena and models play in establishing dynamical chaos in different fields of knowledge. It is *Lorenz model* in hydrodynamics and in the theory of lasers, the *Belousov-Zhabotinsky reaction* in chemistry, *Chua's circuit* in radio-physics, etc.

Chua's circuit has become very popular since the middle of the 1980's [1], [2], because it is, in its physical nature, a rather simple electronic generator of chaos (it consists of four linear elements and one nonlinear circuit element, as shown in Fig. 1).

Chua's circuit is an ideal paradigm for research on chaos by means of both laboratory experiments and computer simulations because it admits an adequate modeling via the language of differential equations. In the simplest case, these equations are written in the dimensionless form

$$\begin{aligned} \dot{x} &= \alpha(y - h(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y \end{aligned} \tag{1}$$

where the nonlinear function $h(x)$ has the form

$$h(x) = m_1 x + (m_0 - m_1)\frac{|x+1| - |x-1|}{2}.$$

The main reasons why Chua's circuit is a subject of interest not only in engineering, but in other disciplines as well, are the following:

1) Chua's circuit exhibits a number of distinct routes to chaos, e.g., transition to chaos through a period-doubling cascade, through the breakdown of an invariant torus, etc., which makes the study of Chua's circuit a rather universal problem.
2) Chua's circuit exhibits a chaotic attractor called the "double-scroll Chua's attractor." It appears at a conjunction of a pair of nonsymmetric spiral Chua's attractors. Three equilibrium states of a saddle-focus type are visible in this attractor, which indicates that the double-scroll Chua's attractor is *multistructural,* which is in sharp distinction with other known attractors of 3-D systems.

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3) Chua's equations (1) are rather "close" (in the sense that the bifurcation portraits are "close") to the equations defining a 3-D normal form for bifurcations of an *equilibrium state* with *three zero characteristic exponents* (for the case with additional symmetry) and that of a *periodic orbit* with *three multipliers* equal to -1.
4) In their mathematical nature, the attractors that occur in Chua's circuit are new and essentially more complicated objects than it seemed before. This conclusion is based on new subtle results on systems with homoclinic tangencies and homoclinic loops of a saddle focus [3], [4].

II. Rigorous Results

The chaotic nature of the double-scroll Chua's attractor was proved by Chua *et al.* [5] by establishing the existence of a homoclinic loop of the saddle focus at the origin and by applying the Shil'nikov theorem. Another proof of the chaotic nature of Chua's circuit, which also makes use of the Shil'nikov theorem, is given by Silva [6]. This theorem asserts that if the 3-D system

$$\begin{aligned}\dot{x} &= \rho x - \omega y + P(x, y, z)\\ \dot{y} &= \omega x + \rho y + Q(x, y, z)\\ \dot{z} &= \lambda z + R(x, y, z)\end{aligned}$$

(where $P.Q, R$ are smooth functions vanishing at the origin along with their derivatives, $\rho < 0, \lambda > 0, \omega \neq 0$) has a *homoclinic loop*, then provided that

$$\rho + \lambda > 0 \tag{2}$$

the Poincaré map on a cross-section transverse to the loop has an infinite number of Smale's horseshoes. It is also important that under small variations of the system, a large number of the horseshoes are preserved.

The nonwandering sets lying near the homoclinic loop of a saddle focus are locally unstable. Therefore, these structures must belong to the attractor in order for the system to exhibit chaotic behaviors. In [5] and [6], parameters were found where this indeed takes place. Actually, the presence of a saddle-focus homoclinic loop in Chua's circuit does *not* guarantee that the double-scroll Chua's attractor is a classical strange attractor having well-understood properties, e.g., sensitive dependence on initial conditions, transitivity, and so on. In fact, the above homoclinic loop *cannot* occur in such typical strange attractors: It was shown in [3] that if condition (2) and the following condition

$$2\rho + \lambda < 0$$

are satisfied, then there exist an infinite number of stable periodic orbits that are dense on the bifurcation surfaces of 3-D systems with saddle-focus homoclinic loops. Furthermore, in any neighborhood of such a surface, the so-called Newhouse regions exist, where systems with infinitely many stable periodic orbits are dense.

III. Future Problems

For the Chua's equations (1), the above observation corresponds to the possibility of parameter values for which infinitely many stability windows exist, which implies a sensitive dependence of the structure of the attractor on small variations of parameters. Besides, as was shown in [4], systems with infinitely many structurally unstable periodic orbits of any degree of degeneracy are dense in the Newhouse regions. Therefore, a "complete description" of the dynamics and bifurcations in the Chua's equations is impossible, as it is for many other models.

The main reason for such a complicated behavior of orbits in the attractors observed in Chua's circuit is connected with the fact that either the attractor itself, or an attractor of a nearby system, contains structurally unstable Poincaré homoclinic orbits, i.e., orbits that arise from the tangency of the stable and the unstable manifolds of some saddle periodic orbit (cycle). If the inequality $|\lambda\gamma| < 1$ is fulfilled where λ and γ are multipliers of the cycle, then the attractor contains stable periodic orbits as a rule. Such an attractor differs essentially from the hyperbolic and the Lorenz attractors. The latter admits the introduction of reasonable ("physical") invariant measures. This makes it possible to study its chaotic behavior by means of statistical methods. In particular, it provides a rigorous foundation for studying such a characteristic of the attractors as Lyapunov exponents. Therefore, hyperbolic and Lorenz attractors are called *stochastic*. On the other hand, the attractors we discuss here hardly admit the introduction of an invariant measure. We called such attractors *quasistochastic* or *quasiattractor* [7], [8]. In our opinion, the reasons formulated above make it natural for us to add some small noise in studying quasiattractors. It is well known that noise is unavoidable both in physical experiments and in computer simulations (due to round-off errors). An explicit introduction of noise could spread stable periodic orbits with long periods and thin basins as well as structurally unstable periodic orbits.

Because they are closely related to the study of homoclinic tangencies, an extension of the results pointed out above for the multidimensional case appears to be rather nontrivial and provides us with opportunities to discover many essentially new effects. Concerning those generalizations of Chua's circuits, which are described by equations of dimension greater than three, the multidimensional theory predicts the following phenomenon: Together, with a "large" attractor, stability windows that exhibit not only periodic and quasiperiodic orbits, but also "small" strange attractors, can exist [9], [10]. These can be attractors of a very different nature, for instance, attractors similar to both the Lorenz attractor and to the double-scroll Chua's attractor. Finally, we remark that there are still many unsolved mathematical problems associated not only with Chua's circuit but also with its globally unfolded canonical circuit [11] and its higher dimensional generalizations, e.g., 1-D chains and 2-D or 3-D arrays of such circuits.

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