Chaos in Some 1-D Discontinuous Maps that Appear in the Analysis of Electrical Circuits

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***Abstract*— Several representative examples of nonlinear electronic circuits modeled by discontinuous 1-dimensional maps, including the 1-D maps derived from Chua's circuit, are reviewed. Although very little general results are presently available for studying the chaotic dynamics of such 1-D maps, an important subclass C where useful properties are known is identified and reviewed. This subclass is characterized by monotonic expansive maps within each continuous subinterval, and where the map assumes at each discontinuity point a left and a right limit equal in value to the boundary (end points) of the defining interval I. The main property characterizing discontinuous maps belonging to class C is that they possess a "good" invariant measure, which can be translated roughly by saying the associated chaotic attractor can be proved rigorously to be ergodic.**

I. Introduction

When investigating nonlinear electrical circuits, we find many examples where the dynamics of such circuits can be described by a *discontinuous* 1-D map derived from the circuit either numerically, or, on rare occasions, analytically. A very large variety of such discontinuous maps has been derived, and not a few mathematical problems relating to these maps remain unsolved.

In the first part of this paper we give examples of some nonlinear circuits and their corresponding 1-D discontinuous maps. Then we dwell upon one rather simple class of such maps for which many general properties can be established. This is the class of *discontinuous* 1-D maps which are *locally expansive* at each point in the domain where the map is given. These maps are characterized by an extreme sensitive dependence of the trajectories on initial conditions. This class of maps possesses some interesting statistical properties, in particular, strong mixing of trajectories, and consequently, various methods from the probability theory can be used for their investigations.

This paper is written mainly for engineers and scientists who employ mathematical methods of nonlinear analysis for research of real physical systems. Our aim is to present a collection of properties and results which are applicable to the above class of discontinuous maps. We will illustrate these results with examples wherever appropriate, but no proofs will be given. Instead, we will include a bibliography showing where these results can be studied in more detail [1]–[27].

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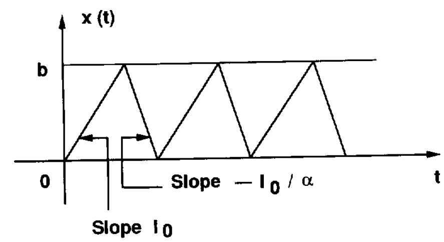


Fig. 1. Output $x(t)$ increases until $x(t)= b$, then decreases until $x(t) = a$ (drawn with $a = 0$), then repeats periodically.

II. Examples of Discontinuous 1-D Maps

Example 1. Triggered Astable Oscillator: First, we will describe how threshold synchronization works. Suppose we have a system that produces a steadily rising output $x(t)$ until some upper threshold b is reached, then produces a steadily falling output until some lower threshold a is reached, and then restarts with a rising output, as in Fig. 1, when $a = 0$. We will work with linearly rising and falling outputs; thus

$$\dot{x} = \begin{cases} I_0 & \text{if } \dot{x} > 0 \text{ }, x \le b \\ -\frac{I_0}{\alpha} & \text{if } \dot{x} < 0 \text{ }, x \ge a \end{cases}$$

where $a < b$. Let us add a narrow periodic pulse $d(t)$ to the output, with period slightly shorter than the natural period of the oscillator. The equation is now

$$\dot{x}(t^+) = \begin{cases} I_0 & \text{if } x(t) + d(t) \le 0 \text{ }, \\ & \quad \text{or if } \dot{x}(t) > 0 \text{ and } x(t) + d(t) < b \\ -\frac{I_0}{\alpha} & \text{if } x(t) + d(t) \ge 0 \text{ }, \\ & \quad \text{or if } \dot{x}(t) < 0 \text{ and } x(t) + d(t) > a \end{cases}$$

where t^+ denotes $\lim_{\varepsilon \to 0}(t + \varepsilon)$.

This elementary mathematical model is an extremely accurate description of the behavior of many circuits; an example is the astable multivibrator, which is given in Fig. 2. It contains a piecewise-linear resistor with the characteristic of Fig. 3. Here the nonlinear resistor charges capacitor C_0 with current I_0 until it reaches point $C(v_R = b)$. An instantaneous transition from C to A is assumed to occur at this point and thereafter resistor R begins to discharge C_0 with a current equal to $-I_0/\alpha$. When it reaches $B(v_R = 0)$ another instantaneous transition from B to D is assumed to occur and R starts charging C_0 again.

The assumption that an instantaneous jump from C to A (and from B to D) occurs is called the *jump postulate* or *phenomenon* in circuit theory [28], and often represents a very realistic model of experimentally observed phenomena. Each boundary point where the jump takes place is called an *impasse* *point* [28], and a comprehensive theory has been developed to show how *impasse point* occurs naturally as a result of idealizations resulting from setting some small but essential circuit parameters (called parasitics) to zero [29]–[31]. Most authors who are unaware of this theory tend to call such double jump phenomenon a *binary hysteresis* [32], [33]. A deeper understanding of what is really taking place will reveal that the phenomenon is not *hysteretic* at all—at least in the usual sense of hysteresis, but in fact is a natural consequence of impasse points, which occurs because of overidealization, thereby preventing the possibility for writing state equations [34].

The model of an astable oscillator enables us to study the effect when it is triggered by external narrow pulses, as shown in Fig. 4. We assume the pulse is so narrow (compared to its period) that we can write (see Fig. 5)

$$d(t) = \begin{cases} c & t = np \\ 0 & \text{otherwise} \end{cases}$$

Let q be the period of the free-running multivibrator. Then, it is shown in [35] that the following 1-D map is the exact description of the behavior of the circuit: $\tau_{n+1} = 1 - (\alpha\tau_n + \beta)\text{mod } 1$, where β is equal to some constant which does not affect the qualitative behavior of the circuit.

An example of this map for $\alpha > 1$ and $\beta = 0$ is shown in Fig. 6.

Example 2. Driven Oscillator with Limit Cycle: As a prototype 2–D nonlinear oscillator with a stable limit cycle we take [36]

$$\dot{r} = sr(1 - r^2)$$
$$\dot{\theta} = 1$$

(in a properly scaled polar coordinate), where s is a parameter (it is a measure of the inverse relaxation time for perturbation of the limit cycle $r = 1$). The oscillator is subjected to a periodic force in the x direction:

$$\dot{x} = sx(1 - x^2 - y^2) - y + 2\alpha \sum_{-\infty}^{+\infty} \delta(t - 2\pi n\beta)$$
$$\dot{y} = x + sy(1 - x^2 - y^2)$$

with $\beta > 0$, and $x = r \;\; cos \;\; \theta, y = r \;\; sin \;\; \theta$, where $\delta(\cdot)$ denotes the delta (impluse) function. The 2-D Poincare map for this system is exactly solvable, giving

$$\left.\begin{aligned} r_{n+1} &= \{[2\alpha + r_n^* cos(\theta_n + 2\pi\beta)]^2 \\ &\quad + [r_n^* sin(\theta + 2\pi\beta)]^2\}^{1/2} \\ \theta_{n+1} &= tan^{-1}\left[\frac{r_n^* sin(\theta n + 2\pi\beta)}{2\alpha + r_n^* cos(\theta n + 2\pi\beta)}\right] \end{aligned}\right\}$$

where

$$r_n^* = \frac{r_n}{[r_n^2 + (1 - r_n^2)e^{-4\pi r}]^{1/2}}$$
$$\gamma = s\beta \tag{1}$$

and r_n, θ_n are the values of r, θ immediately after the nth "kick" at $t = 2\pi n\beta$.

If we let $\gamma \to \infty$, the fast-relaxation limit, we obtain $r_n^* = 1$. This reflects the obvious result that the oscillator returns to its limit cycle before the next "kick". In this limit we obtain a 1–D map:

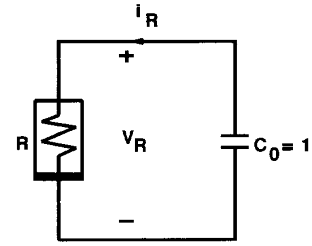


Fig. 2. Circuit model of the astable oscillator.

$$\theta_{n+1} = tan^{-1}\left[\frac{sin(\theta_n + 2\pi\beta)}{2\alpha + cos(\theta_n + 2\pi\beta)}\right].$$

We note that in the case of finite damping with $\gamma \leq -1, exp(-4\pi\gamma) < 10^{-5}$, the return map (1) is almost 1-D.

Two maps corresponding to α = 0.3, and 0.5 are shown in Fig. 7.

We stress that this system (and consequently the 1-D map) is a prototype nonlinear oscillator since the structure of the parameter space (α, β) is qualitatively similar to the following cases: Brusselators [37], electronic oscillators tuned with nonlinear elements and having its frequency synchronized by means of an external impulsive periodic signal [38], and forced Van der Pol oscillators [39].

Example 3. Chua's Circuit: One-dimensional approximate maps for piecewise-linear autonomous systems have been consrtucted and analysed by Sparrow [40], Brockett [41], Chua *et al.* [42], Ogorzalek [43]. In all these cases, the basic assumption is that there exists "strong stable foliations" in the the neighbourhood of some fixed point. The 1–D maps of the half-line BA_∞ (see notation in [42]) into itself, and P^+N_∞ into itself are presented in Figs. 8 and 9; see [44].

Recently, Brown [45] has shown that the basic chaotic dynamics of Chua's circuit can be accurately modeled by a broad class of discontinuous 1-D maps. Depending on the circuit parameters, these 1-D maps [46] can be quite simple (called class $\mathcal{C}$ below) as shown in Fig. 10, whereas others can be rather complicated, such as those shown in Figs. 11 and 12.

Example 4. Second-Order Oscillator with Impasse Points: First, we outline how a second-order oscillator with impasse points worked. In Fig. 13, we draw the phase space of a typical oscillator having binary hysteresis; it contains two parallel invariant half-planes which overlap in the strip, say $-1 < z < 1$. Each half-plane represents one of the phase space of a linear unstable degree-two oscillator. When a trajectory of one of these linear oscillators reaches the boundary of its half-plane (the crossing of a point from one half-plane to the other takes place on the lines $z = 1$ and $z = -1$), the trajectory jumps to the other half-plane, by assumption, as shown in Fig. 13.

There are many examples of oscillators with impasse points (see [32], [33], [47], [48], and [49]). One such example is presented by Pikovskii and Rabinovich [47]. In dimensionless

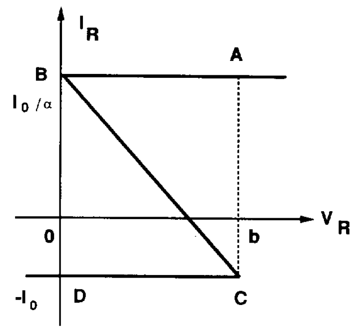


Fig. 3. The $v_R - i_R$ characteristic of the nonlinear resistor in Fig. 2.

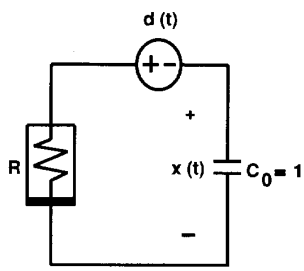


Fig. 4. Circuit model of the triggered astable oscillator.

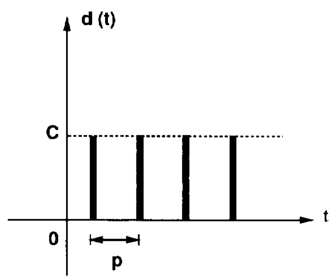


Fig. 5. The synchronization signal.

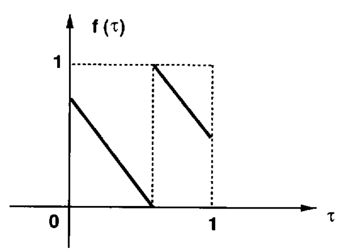


Fig. 6. The 1-D map for the circuit shown in Fig. 4.

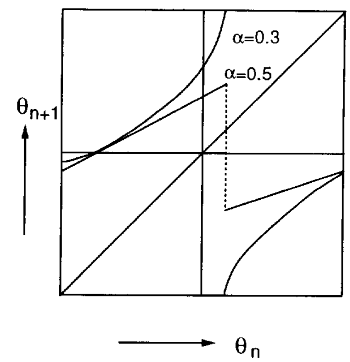


Fig. 7. The 1-D map for the driven oscillator with limit cycle in the fast-relaxation limit.

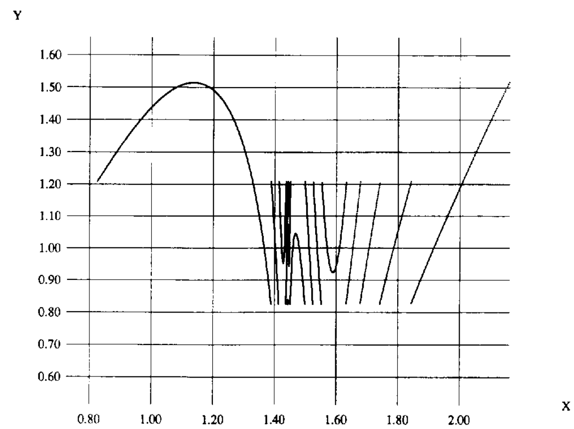


Fig. 8. The 1-D map of the half-line BA_∞ (see notation in [42]).

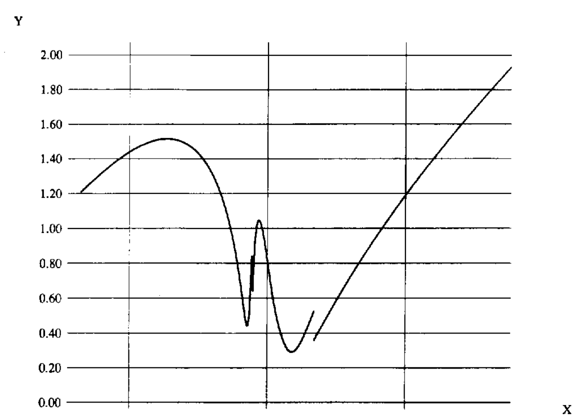


Fig. 9. The 1-D map of the half-line P^+N_∞ (see notation in [42]).

variables, their circuit (Fig. 14) is described by the equations

$$\left.\begin{aligned} \dot{x} &= y - \delta z \\ \dot{y} &= -x + 2\gamma y + \alpha z \\ \mu\dot{z} &= x - f(z) \end{aligned}\right\} \qquad (2)$$

where $f(z)$ is the idealized (dotted) characteristic shown in Fig. 15. As $\mu \rightarrow 0$ all motion of the system can be separated into a fast regime (along the lines x= const and y = const), and a slow regime (in the planes $z = -1, x < 1$ and $z = 1, x > -1$). If $\mu = 0$, we have a circuit with a line of impasse points because $f^{-1}(x)$ is a triple-valued function for $|x| < 1$.

Let us define S^- as the half-line located at $x = -1, z = -1, y > -\delta$, and S^+ as the half-line located at $x = 1, z =$

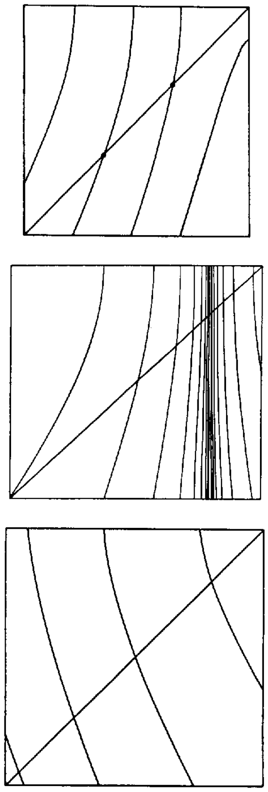


Fig. 10. Some class C 1-D maps from the generalization of Chua's equations.

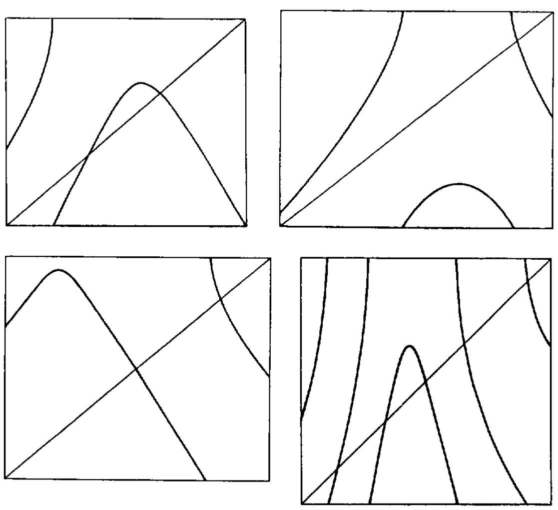


Fig. 11. Some 1-D maps with single hump from the generalization of Chua's equations.

$1, y > \delta$. The dynamics of the system (2) reduces to the analysis of the Poincare map of the set $S = S^{+} \cup S^{-}$ into itself. All trajectories beginning and ending in S can be divided into two groups:

1) Trajectories lying in only one half-plane:

$$\tilde{S} = T_1 S = e^{2\pi k} S.$$

2) Trajectories that cross from one half-plane to the other:

$$\tilde{S} = T_2 S$$

where T_2 is given parametrically by

$$S = \frac{\omega e^{-k\tau}}{sin \ \tau}$$
$$\tilde{S} = 2\delta - \frac{\omega(cos \ \tau + ksin \ \tau)}{sin\tau}$$
$$\omega = (1 - \gamma^2)^{1/2}, k = \frac{\gamma}{\omega}.$$

The Poincare map associated with the above 1-D map is discontinuous, as shown in Fig. 16.

Example 5. Switching Circuit: Consider the circuit shown in Fig. 17 [48]. In this figure, g_1 is a linear negative conductance. For $t \geq 0$, S is switched on or off when a trajectory intersects the ϕ axis on the $\phi - v$ plane as follows:

a) S is closed at the moment when $\phi_t < \phi$ and $v = 0$, where ϕ_t is a threshold of ϕ such that $\phi_t > 0$ and $\phi_t > log_2 v$.
b) S is opened at the moment when $\phi_t \geq \phi$ and $v = 0$. For this map, the Poincare map can be constructed exactly as (see Fig. 18).

$$F : [0, 1] \rightarrow [0, 1]$$

$$F(x) = \begin{cases} a(x - D) + 1, & 0 < x \leq D \\ b(x - D), & D < x \leq 1 \end{cases}$$

Example 6. $\Sigma\Delta$ Modulator: The structure of the ideal single-loop $\Sigma\Delta$ modulator is as shown in Fig. 19 [50]. It consists of a discrete time integrator with a quantizer in a feedback loop. The only nonlinear element in the modulator is the 1-b quantizer, whose output is 1 when its input is ≥ 0 and -1 when its input is < 0. Under the assumption that the input to the modulator is constant, the system is described by the first-order difference equation [50]:

$$u_{n+1} = u_n + x - sgn(u_n)$$

where x is the input to the modulator, and u_n is the quantizer input. The map $u_n \rightarrow u_{n+1}$ is plotted in Fig. 20.

All of the 1-D maps given above are discontinuous. The class of discontinuous maps, even with only a finite number of discontinuity points, is essentially larger than the class of continuous 1-D maps. We wish to point out first that there are still many unsolved complicated problems for continuous and even for smooth maps (for example, existence of invariant measures and their properties, investigation of the correlation between trajectories etc.). All the more, the same is true as far as discontinuous maps are concerned. It is difficult if not impossible to develop a general theory of discontinuous maps because there exist too many variations and pathological situations that warrant a special treatment. A more or less "rich" theory can be developed only for some relatively narrow classes of discontinuous maps.

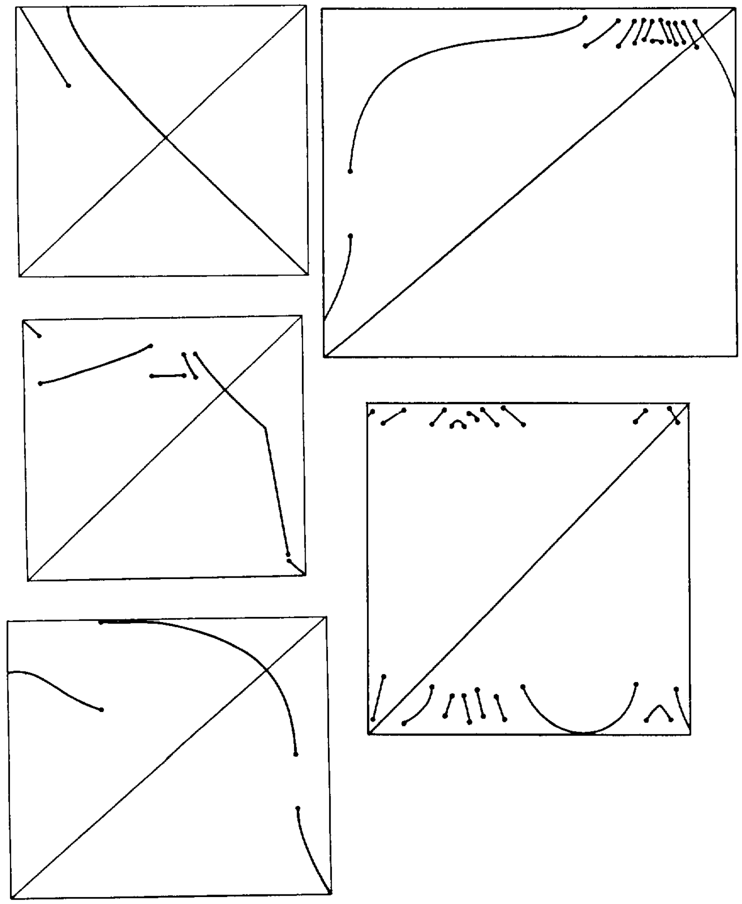


Fig. 12. Some complicated 1-D maps from the generalization of Chua' equations.

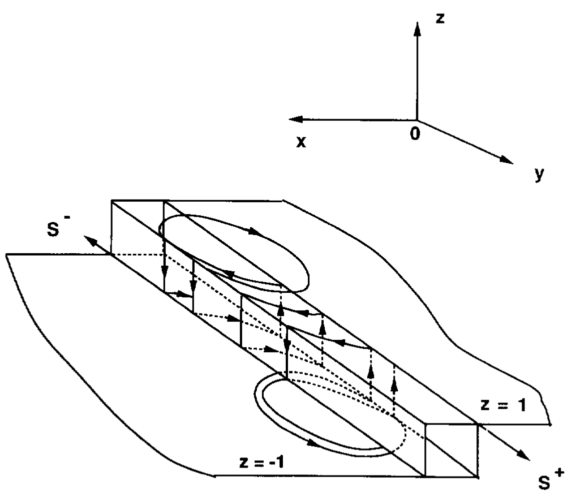


Fig. 13. The phase space of the second-order oscillator with impasse points.

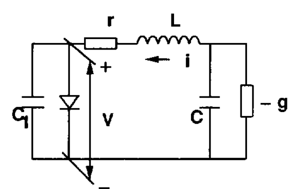


Fig. 14. The circuit model of the second-order oscillator with impasse points.

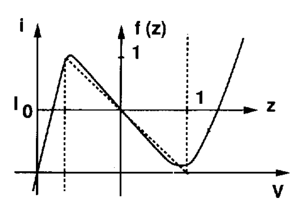


Fig. 15. The idealized (dotted) characteristic of the tunnel diode.

III. Class C Discontinuous 1-D Maps

Although the above maps are discontinuous, some of them have discontinuity points which share a useful common property (namely, the limit values at the discontinuity points belong to the boundary) which makes them more similar to continuous maps, however, not of an interval but of a circle. If our map f is given on some interval $I = [a, b]$ and if $z_1, ..., z_r, r < \infty$, are discontinuity points, then for this class of

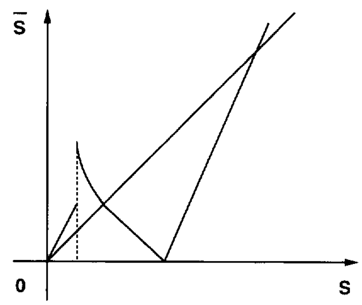


Fig. 16. The 1-D map from the circuit shown in Fig. 14.

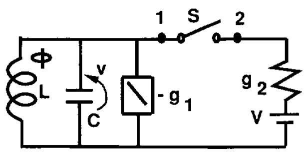


Fig. 17. The circuit model from [49].

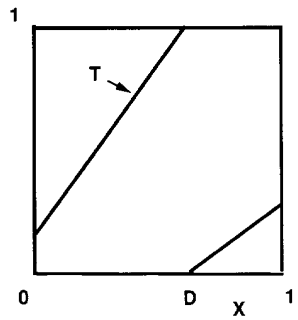


Fig. 18. The 1-D map obtained from the circuit shown in Fig. 17.

maps, we have $\lim_{x\to z_i\pm 0} f(x) \in \partial I = \{a, b\}, i = 1, ..., r$ (and moreover, if $\lim_{x\to z_i - 0} f(x) = a$, then $\lim_{x\to z_i+0} f(x) = b$, and if $\lim_{x\to z_i-0} f(x) = b$, then $\lim_{x\to z_i+0} f(x) = a$). This means that if we identify the points a and b with each other, then the map f can be regarded as a map $\tilde{f}$ of the circle, and the map $\tilde{f}$ is continuous everywhere except, very likely, at the point a, because $\lim_{x\to a+0} \tilde{f}(x) = f(a)$ and $\lim_{x\to a-0} \tilde{f}(x) = f(b)$. Furthermore, the peculiarity of the map is that on the intervals of continuity (z_i, z_{i+1}), $i = 0,1,2,...,n$, where $z_0 = a$ and $z_{n+1} = b$, the map f is either monotonic (increases or decreases) or unimodal (i.e., there exists a critical point $c_i \in (z_i, z_{i+1})$ and the function f increases on one of the intervals (z_i, c_i) and (c_i, z_{i+1}) and decreases on the other).

The class of such discontinuous maps is still very large. In particular, it contains the class of continuous unimodal maps which has been intensively investigated for the last 20 years, and now a rich and meaningful theory has been developed for it. We shall restrict ourself in this paper to considering an even more narrow class of discontinuous maps for which the map is monotonic on each interval of continuity, and moreover, it is locally expansive in each point.

Thus, let f be a map of the interval $I = [a, b]$ into itself, which satisfies the following conditions:

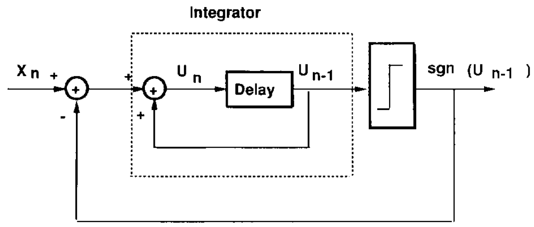


Fig. 19. The ideal single-loop $\Sigma\Delta$ modulator.

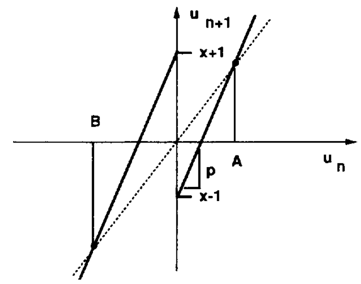


Fig. 20. The 1-D map obtained from the $\Sigma\Delta$ modulator.

1) It is continuous everywhere except at the points $z_1, ..., z_r (a < z_1 < ... < z_r < b, 1 \leq r < \infty)$.
2) It is monotonic on each interval $(z_i, z_{i+1}), i = 0, 1, .., r$, $z_0 = a$, $z_{r+1} = b$.
3) $\lim_{x\to z_i-0} f(x) \neq \lim_{x\to z_i+0} f(x)$ and $\lim_{x\to z_i\pm 0} \cdot f(x) \in \{a, b\}, i = 1, ..., r$.
4) It is expansive with an expansion coefficient $l > 1$; for every point $x \in I \setminus \{z_1, ..., z_r\}$, there exists an interval U_x containing the point x such that $d(f(U)) > l d(U)$ for every interval $U \subset U_x$, where $d(V)$ denotes the length of the interval V.

The last condition, of course, narrows the class of maps under consideration in an essential way. It follows from this condition that the map has the so-called property of "hyperbolicity"(this property was first used in a systematic way by Anosov for the study of dynamical systems on surfaces of constant negative curvature). The notion of hyperbolicity means that the relative positions of trajectories are similar to those near the saddle point of a dynamical system on the plane.

Let $\mathcal{C}$ denote the class of 1-D discontinuous maps satisfying the conditions 1)–4). It is not very difficult to explain a number of important (in particular, as regards to applications) properties of the maps from $\mathcal{C}$. Here we do not formulate the strongest results and do no more than to predict results which can be formulated and understood in a simple way. We will try to demonstrate all the main (from the authors point of view) properties of maps belonging to the class $\mathcal{C}$. Apparently, the most essential property is the existence of an *invariant measure*, which differs only slightly from the Lebesgue measure.

IV. Instability of Trajectories

Each map $f: I \to I$ defines a dynamical system on the interval I. Each point $x_0 \in I$ determines the trajectory of this dynamical system (or the trajectory of the map) $x_0, x_1, ..., x_i, ...$, where $x_{i+1} = f(x_i) (= f^{i+1}(x_0)), i = 0, 1, 2, ...$

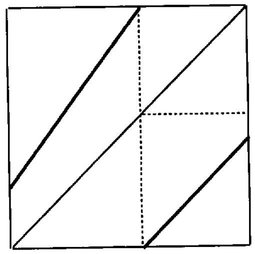


Fig. 21. An example of a 1-D map without fixed points, but possessing a cycle of period 3, and cycles with period 2.

It follows immediately from property 4) that any two trajectories $\{x_i\}_{i=1}^{\infty}$ and $\{x_i^{'}\}_{i=1}^{\infty}$ which are close to each other initially must move away from each other in the course of time. Moreover, at all subsequent moments, the distance between the points x_i and $x_i^{'}$ increases not less than l times: $|x_{i+1} - x_{i+1}^{'}| \geq l|x_i - x_i^{'}|$ (admittedly it is true up to the moment $i = N$, where x_i and $x_i^{'}$ belong to the same interval (z_j, z_{j+1}). This means that for every trajectory, its Lyapunov exponent is equal to or greater than l and all trajectories are unstable in the sense of Lyapunov. Thus, the trajectories of dynamical systems from C exhibit an extreme sensitivity to initial conditions.

V. Periodic Points and Semi-Periodic Intervals

Periodic trajectories hold usually an important place when investigating dynamical systems. In this case, however, all periodic trajectories of any map belonging to C are unstable and are of little importance for understanding the dynamics of maps. Therefore, we restrict ourselves to only a few remarks.

It is easy to show that there exist periodic trajectories with an arbitrarily large period. Since the map is discontinuous it can have, for instance, a cycle with period 3, at the same time (as distinct from continuous maps), it can have neither fixed points nor cycles with period 2 (Fig. 21).

At the same time, if $n \geq 3$ the map has cycles with all periods (in particular, at the interval (z_1, z_3); since $f([z_1, z_2])$ = $f([z_2, z_3]) = I$ we may use the simplest variant of the so-called symbolic dynamics: on the interval $[z_1, z_3]$ there exists an invariant Cantor set $K = \cap_{i=0}^{\infty} f^{-i}([z_1, z_3])$ (here $f^{-i}(y)$ $=\{x \in I / f^i(x) = y\}$) on which the map f is topologically conjugate to the shift map on the space of one-sided sequences of zeros and one's.

The intervals which we shall call semi-periodic are of greater importance for understanding the dynamics than the periodic points. When investigating continuous maps (in particular, in the construction of the so-called spectral decomposition), the periodic intervals are used (J is a periodic interval with period s if $f^s(J) = J$ and $f^i(J) \neq J$ for $0 < i < s$). Maps from C can have no periodic intervals with period $s > 1$, and *semi-periodic intervals*, which will be defined below, are to a certain extent the analog of periodic intervals.

The map represented in Fig. 22 has, for example, two intervals I_1, I_2 such that $f(I_1 \cup I_2) = I_1 \cup I_2$. Although intervals I_1, I_2 are not periodic, their union is periodic. We shall call them semi-periodic; in this case, they form a cycle of length

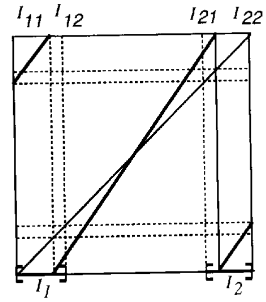


Fig. 22. An example of a 1-D map with two semi-periodic intervals: I_1 and I_2.

2. Each interval I_i, $i = 1, 2$, contains, in turn, two subintervals $I_{i1}, I_{i2}, i = 1, 2$, such that $f(\cup_{i,j=1}^{2} I_{ij}) = \cup_{i,j=1}^{2} I_{ij}$, i.e., the subintervals I_{11}, I_{12}, I_{21}, and I_{22} also form a cycle of length 4 and are semi-periodic.

The importance of such intervals, at least for maps, such as those shown in Fig. 21, is due to the fact that the invariant sets which are formed are stable (i.e., closely spaced trajectories are attracted to them and even fall inside these intervals). Moreover, these invariant sets attract almost all (but not all) trajectories.

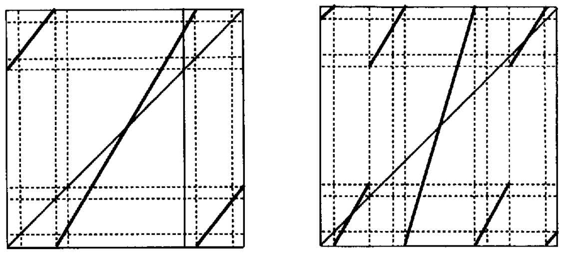
We note that all semi-periodic intervals always have common points with one of the intervals (a, z_1) and (z_n, b).

VI. Trajectory Attractors

In the dynamical system theory for describing asymptotical (when time tends to ∞) behaviour of trajectories ω-limit sets are used (the ω-limit set $\omega(x_0)$ of trajectory $x_0, x_1, x_2, \ldots$ is a set $\{x / \exists i_1 < i_2 < \ldots : x_{i_k} \to x$ when $k \to \infty\}$, i.e., $\omega(x_0) = \cap_{n=0}$ (closure of $\cup_{i=n}^{\infty} f^i(x_0)$). It is advisable, however, to use ω-limit sets only for sets consisting of unstable trajectories, in particular, for maps with sensitive dependence on initial conditions (which we have). Here, it is expedient to consider, along with trajectories and their ω-limit sets, the so-called *prolongations of the trajectories* $pr(x_0) = \cap_{\varepsilon > 0}$ (closure of $\cup_{i=1}^{\infty} f^i(U_\varepsilon(x_0))$), where $U_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) \cap I$ and, respectively, ω-prolongations ω-pr(x_0) $=\cap_{i=1}^{\infty} f^i(\text{pr}(x_0))$. Obviously, we have always $\omega(x_0) \subset \omega$-pr$(x_0)$. Both ω-limit set and ω-prolongation are invariant closed sets. The trajectory passing through the point x_0 is stable in the sense of Lyapunov if and only if $\omega(x_0) = \omega$-pr(x_0). If we use the computer to observe the ω-limit set of some trajectory on the screen it will mean that in reality because of the finite accuracy of our computations, we shall actually observe the ω-prologation on the screen.

For our map, the ω-prolongation appears to be one and the same set for all (or almost all) trajectories (i.e., it does not depend on the initial point x_0) and consists of either one interval (the whole I) or several subintervals. These subintervals are semi-periodic and form an invariant set which can not contain proper invariant subsets consisting of semi-periodic intervals.

Example 7: Let f_λ be the map with a constant slope $\lambda = 2^{1/k}, k \leq 2$ (and, therefore, $r = 2$). Let $z_1 = 1 - z_2$ (Fig. 23).



(a) (b)

Fig. 23. (a) A 1-D map with a constant slope $2^{\frac{1}{k}}$, $k \leq 2$. (b) The second iteration of the 1-D map from (a).

For any trajectory of this map, the ω-prologartion consists of 2^{k-1} intervals (except for at most a countable number of trajectories passing through periodic points that do not belong to these intervals, there is a finite number of such periodic points).

At present, the concept of an attractor is widely used in the dynamical system theory. We can talk about the attractor of a particular trajectory passing, say, through the point x_0, and in this case, we can take as the attractor either ω-pr (x_0), or $\omega(x_0)$, or even a smaller set: the statistical limit set (in any but fixed neighborhood of which the points $f^i(x_0)$ appear with a frequency approaching 1 when $i \to \infty$). From the "practical" viewpoint, we should choose the ω-prologation, and then, as explained above, the attractor will be the same for almost all trajectories (although for almost all trajectories, both the ω-limit set and the statistical limit set will coincide with the ω-prologation; this is true for the map considered here as well as for a very broad class of continuous maps).

VII. Global Attractor

When we talk about an attractor, we have most often in mind a global attractor of the dynamical system, i.e., an invariant and closed set that attracts all or almost all (in some sense) trajectories. Different authors use different definitions of attractor as the least set containing ω-limit sets of almost all trajectories.

In our case, taking into account the above discussions, the attractor of a 1-D map (in any natural sense) consists of one or several (semi-periodic) intervals.

Denote the attractor by A. It is easily seen that $A \subset f([a, z_1] \cup [z_r, b])$. Some properties of the map on the attractor include the following:

1) There exists a strong expansion: for any open set u(e.g., open interval) belonging to A there will be found an integer m such that $f^i(u) = A$ when $i \leq m$.
2) There exist dense trajectories everywhere on A.
3) Periodic trajectories are dense on A.

VIII. Invariant Measure

It seems that the most important property of the maps from $\mathcal{C}$ is the fact that the dynamical systems has a "good" invariant measure. The invariance of measure μ means (by definition) that $\mu(B) = \mu(f^{-1}(B))$ where B is an arbitrary μ-measurable set. The word "good" means that this measure differs not very strongly from the usual Lebesgue measure, namely, it is absolutely continuous with respect to the Lebesgue measure (i.e., the μ-measure of any set which has zero Lebesgue measure is also equal to 0).

We can state the following theorem concerning the existence of an invariant measure for maps belonging to $\mathcal{C}$ [15] if $f \in C^2((z_i, z_{i+1}))$, $i = 0, 1, 2, ..., r$, and $|f'| > 1$ on $I \setminus \{z_1, ..., z_r\}$. Then, the map f has an invariant measure that is absolutely continuous with respect to Lebesgue measure.

We shall always denote this invariant measure by μ. It may be shown that the measure μ is the only "good" measure and, moreover, is *ergodic*. The ergodicity means that the μ-measure of any invariant closed set is either equal to 0, or the μ-measure of the whole space.

The measure μ is concentrated on the attractor A $(i.e., \mu(A) = \mu(I))$, which consists, as we know, of one or several intervals.

The importance of this invariant measure is explained by the famous Birkhoff- Khinchin theorem (which holds for any invariant measure). This theorem, in particular, states that $\frac{1}{N}\sum_{i=0}^{N} \mathcal{X}_B(f^i(x_0)) \longrightarrow \mu(B) as N \to \infty$ for each μ-measurable set B, and for almost every (by measure μ) point x_0, where

$$\mathcal{X}_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

This theorem implies that in order to understand the asymptotical behaviour of trajectories when the μ-measure support has a positive Lebesgue measure (as in our case), it is necessary to use the probability language; if we have some μ-measurable set B, a trajectory starting from almost any point will fall within the set B with frequency that tends to the μ-measure of the set B as time increases. Thus, the invariant measure gives an additional information about the trajectory's behavior.

For example, the map $f : x \mapsto kx + \beta$ (mod 1) with any integer $k \neq 0, \pm 1$ and with any β, $0 \leq \beta < 1$, has the Lebesgue measure as an invariant measure. Thus, if we are interested in knowing how often a trajectory $f^i(x_0)$ will fall, say, into (γ_1, γ_2), then the answer is the following: The frequency of falling into (γ_1, γ_2) tends to $\gamma_2 - \gamma_1$ for almost every (by the Lebesgue measure) point $x_0 \in [0, 1]$.

How can one find the invariant measure for a concrete map from $\mathcal{C}$? The Birkhoff–Khinchin theorem states that almost every trajectory must generate a set of points with this measure: If we have some point x_0 through which such a trajectory passes (and it is almost any point), we should partition our space (the interval I) into a sufficiently large number of boxes—small subsets (subintervals of the lengh $\varepsilon << 1$), then choose some integer $N \gg 1(\gg \frac{1}{\varepsilon})$ and calculate how often the points $f^i(x_0)$ falls into each box where $0 \leq i \leq N$ (i.e., we should construct the histogram of the trajectory). When $N \to \infty$, the frequency of the points $f^i(x_0)$ inside each subset must approach the μ measure of the subset, and when $\varepsilon \to 0$, we shall get the density of the measure μ.

We must emphasize that, in general, not all, but only almost all, points generates a set with the measure μ. For example, no periodic point generates a set with a "good" measure, and in our case (as we had noted above), the periodic points lie everywhere dense on the global attractor A, which is the support of measure μ.

IX. Numerical Orbits and Shadowing Property

Let us focus our next attention to one more property which may or may not be possessed by maps from $\mathcal{C}$. This property should be kept in mind when computing the trajectories.

Since for map $f \in \mathcal{C}$ the coefficient of "expansion"$\geq l > 1$, the length of any small interval J is increased l times under each iteration of the map f: If the length $d(J) = \varepsilon$ then $d(f^n(J)) \geq l^n \varepsilon$ at least for n such that $f^i(J) \cap \{z_1, ..., z_r\} = \phi$ when $1 \leq i \leq n$. This means that in approximately $N \simeq \frac{log\frac{1}{\varepsilon}}{logl}$ steps the length of the interval $f^n(J)$ will be of the order 1 (i.e., of the whole space size). Thus, if the exactness of computations is the order of ε we can say almost nothing about the location of the point $f^n(x_0)$ when $n > N$, if $x_0 \in J$ (i.e., we can not state that the point $f^n(x_0)$ with $n > N$ will be located at the position calculated by a computer.)

Nevertheless, there are dynamical systems that possess the following property: For any ε-trajectory $x_0, x_1, ...$ (i.e., for the numerical trajectory that was calculated with the exactness ε: $|x_{n+1} - f(x_n)| < \varepsilon$ when $n \geq 0$), there exists a true trajectory $x_0^{tr}, x_1^{tr}, ...(x_{n+1}^{tr} = f(x_n^{tr}))$ such that $|x_n - x_n^{tr}| < \delta$ for some small $\delta > 0$ and for all $n \geq 0$. In this case, we say that numerical ε-(pseudo) trajectory is the δ-shadow of a true trajectory, and the dynamical system possesses the shadowing property.

For example, the map $f : x \mapsto 2x \pmod 1$ has the shadowing property. This map can be represented by the formula $f : 0.\alpha_1\alpha_2\alpha_3... \mapsto: 0.\alpha_2\alpha_3...$, where $\alpha_i, i = 1, 2, ...$, is 0 or 1. If the computer can only guarantee $N - 1$ exact digits, then for $x_0 = 0.\beta_1\beta_2...\beta_N\beta_{N+1}...$, we will obtain

$$\begin{aligned} x_1 &= 0.\beta_2...\beta_N \\ x_2 &= 0.\beta_3...\beta_N\gamma_1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{N-1} &= 0.\beta_N\gamma_1...\gamma_{N-1} \\ x_N &= 0.\gamma_1\gamma_2...\gamma_N, \\ &\quad \text{where } \gamma_i \in \{0,1\} \text{ but does not depend on } \beta_i \\ &\quad \text{for all } i \leq 1 \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Thus, starting from x_N, the numerical ε-trajectory with $\varepsilon = 2^{-N}$ "forgets" the initial point x_0. Nervertheless, there exists a true trajectory with the initial point $x_0^{tr} = 0.\beta_1...\beta_N\gamma_1\gamma_2...$, which is ε close to the numerical trajectory.

However, not every map from $\mathcal{C}$ possesses the shadowing property. Let us consider, for instance, the map $f : x \mapsto 2x + 1/2 \pmod 1$, which can be represented by $f : 0.\alpha_1\alpha_2\alpha_3... \to 0.\bar{\alpha}_2\alpha_3...$, where $\alpha_i \in \{0, 1\}$ and $\bar{\alpha}_2 = 1 - \alpha_2$. For arbitrary integers $m \geq k > 1$, the sequence of points $x_0 = 0$, $x_i = 0.0\underbrace{1...1}_{m-i}$, $i = 1, 2, ..., m - k + 2$, is a piece of the ε-trajectory with $\varepsilon = 2^{-m}$(because $x_1 = f(x_0) - \varepsilon$, $x_{i+1} = f(x_i)$ when $i > 0$). However, if $0 \leq x_0^{tr} \leq \delta = 2^{-k}$ then

$$\max_{0 \leq i \leq m-k+2} |x_i - f^i(x_0^{tr})| \geq 0.\underbrace{0...0}_{k-1}1 > 2^{-k} = \delta$$

i.e., the δ-shadow of the ε-trajectory $x_0, x_1, x_2, ...$ contains no true trajectory (for any x_i when $i > m - k + 2$).

Recently, a large number of papers has appeared that deals with the conditions that should be satisfied for a dynamical system to have the shadowing property, including an estimate of the quantity ε, δ, etc. These papers are mainly devoted to continuous and smooth dynamical system. The first work in this direction is due to Anosov[1].

Investigations on the shadowing property for discontinuous maps are virtually nonexisting.

X. Topological Entropy

So far, we have not mentioned about such notion as topological entropy which is widely used now in the dynamical system theory. The topological entropy tells us about the diversity of the trajectory behaviour (more exactly, to what extent the sets of the trajectory parts of finite (time) length vary with an increase of this length). It is known, for instance, that for continuous 1-D maps the topological entropy is positive if and only if the map has a cycle of the period $\neq 2^i, i = 0, 1, 2, ...$ This means, in particular, that the attractor of the dynamical system can consist of one periodic trajectory (to which almost all (by Lebesgue measure) trajectories will be attracted), but at the same time, its topological entropy will be positive as, for instance, for the map $x \mapsto \lambda x(1 - x)$ with $\lambda = 3.83$. Here, the topological entropy is positive (equals $\ln\frac{1+\sqrt{5}}{2}$), and almost every point is attracted to the cycle of period 3.

For the maps from $\mathcal{C}$, the *topological entropy always differs from zero*. For $r > 2$, as has been mentioned above, the map on some Cantor set is equivalent to the symbolic dynamical system with $r - 1$ symbols, and therefore, its topological entropy is at least not less than $ln(r - 1)$.

XI. Concluding Remark

We have described the properties of one class of dynamical systerms which are modeled by discontinuous 1-D maps. The main property is the existence of a "good" invariant measure, and, as a corollary, the existence of a *stochastic attractor*. In this case, we should use the probability language to characterize the behaviour of the trajectories. For every subset of the attractor, and for almost every trajectory, we may find (or know) only the probability of falling into a prescribed subset of a point moving on this trajectory as time tends to infinity. This class of maps always has strong temporal chaos.

It is not possible at this point in time to present a detailed analysis of all properties mentioned in this paper, or for the analysis of other important properties, such as bifurcations, special properties of invariant measures, and in particular, correlation properties of trajectories.

We have considered only one narrow class of discontinuous maps in this paper. What can we say about other classes of discontinuous maps? Discontinuous maps is a very complicated research subject and we can obtain useful and interesting results only for various special classes of maps.

Of course, discontinuous maps of other classes can have essentially different properties. For instance, maps that violate conditions 2 and 4 cannot have chaotic behavior. They can have attracting cycles that attract almost all trajectories, and therefore, almost all trajectories are stable in the sense of Lyapunov. In this case, there does not exist a good invariant measure; however, as for maps belonging to class $\mathcal{C}$, cycles with unboundedly large periods exist, and their topological entropy is positive.

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