Experimental observation on the effect of coupling on different synchronization phenomena in coupled nonidentical Chua’s oscillators

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Experimental results are presented on the effect of coupling on synchronization of two coupled nonidentical Chua’s oscillators. Two oscillators are coupled in unidirectional drive response mode. The driver is always kept in chaotic (single scroll, double scroll) state while the response oscillator is kept in various dynamical states as point attractor, single scroll periodic, chaotic and double scroll. The strength of coupling plays a crucial role on synchronization between the coupled oscillators. With decreasing coupling strength, two routes of transitions, one route is through lag and intermittent lag synchronization and another one is through intermittency have been observed in single scroll cases. But the situations are slightly different when the driver is double scroll chaotic. Lag synchronization and intermittent lag synchronization regimes are present in double scroll situation, but an intermediate intermittency regime between ILS and PS has also been observed.

Synchronization of chaotic systems draws attention of researchers in many areas like secure communication, neuroscience, laser, chemical oscillations since natural systems are nonlinear and they behave chaotically in a specific parameter domain. Enormous numbers of studies on synchronization of the coupled chaotic system have been reported to date, but a unified definition is yet to be reached. Several regimes of chaotic synchronization, like complete synchronization, lag synchronization, intermittent lag synchronization, phase synchronization, have been observed in both theoretical and experimental studies on different systems. All such synchronization phenomena and their stability depend upon the nature and strength of coupling, and also both on system noise and external noise. In this paper, experimental results on synchronization of coupled nonidentical Chua’s oscillators are reported. Two nonidentical oscillators are coupled in unidirectional drive response mode. Different oscillatory states of response oscillator (point attractor, period-I, chaotic) are taken into consideration while the driver is always chaotic. The main thrust of this work is on how complete synchronization changes to phase synchronization as the strength of the coupling is gradually decreased and ultimately the coupled system becomes completely asynchronous. Two routes of transitions, one route is through lag and intermittent lag synchronization and another one is through intermittency have been observed in single scroll cases. But the situations are slightly different when the driver is double scroll chaotic. Lag synchronization and intermittent lag synchronization regimes are present in double scroll situation, but an intermediate intermittency regime between ILS and PS has also been observed.

I. INTRODUCTION

Chaotic synchronization\(^1\) of coupled systems has now become a major topic of interest in the study of nonlinear dynamics. Coupling plays an important role in such synchronization of coupled systems. In fact, one can distinguish several regimes of chaotic synchronization between coupled systems depending upon the nature and strength of coupling. For identical chaotic systems, in presence of strong interaction, the amplitude and phase of two systems may coincide\(^2\) after a certain elapse of time. Thus the state vectors \(\mathbf{x}_1(t), \mathbf{x}_2(t)\) of coupled systems coincide to each other, \(\mathbf{x}_1(t) = \mathbf{x}_2(t)\) as \(t \to \infty\), which is called complete synchronization (CS). Since two systems can never be identical in nature, synchronization has also been tried with two slightly mismatched systems. In such a case, a functional relationship between the states of response and drive system emerges as \(\mathbf{x}_1(t) = F(\mathbf{x}_2(t))\), \(t \to \infty\), which is referred to as generalized synchronization (GS).\(^3\) Various other types of synchronization in coupled chaotic systems have been observed later in the weaker coupling limit, namely, lag synchronization (LS),\(^4–7\) intermittent lag synchronization (ILS),\(^4–7\) and phase synchronization (PS).\(^8\) Transition route from CS or GS to PS through subsequent synchronization stages of ILS and LS has been observed\(^6,7\) with decreasing coupling strength. Another route from CS to PS through intermittency has been reported\(^9\) recently. In phase synchronization (PS), the phases of two chaotic oscillators are locked to each other, i.e., the phases \(\phi_{1,2}\) of the driver and response satisfy the relation \(|n \phi_1 - m \phi_2| < \text{constant}\) (\(n\) and \(m\) are integers), while the am-

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plitudes remain uncorrelated. In simplest case of 1:1 phase locking \((m=n=1)\), the phase difference \(|\phi_1 - \phi_2| < \text{const}\) and remains bounded in time. On the other hand, LS is characterized by the coincidence of the state of drive system with the state of response system shifted in time, i.e., \(x_1(t - \tau) = x_2(t)\), where \(\tau\) is the lag time. Recently, theoretical studies\(^5\) are done to characterize the inherent details of ILS phenomenon, which is found to occur between PS and LS regimes. ILS has been characterized in terms of the existence of a set of lag times \(\tau_n (n = 0, 1, 2, 3, ...)\) such that the system always verifies \(x_1(t - \tau_n) = x_2(t)\) for a given \(n\), which occurs due to local instabilities and leads to occasional bursts of nonsynchronous behavior. In this regime, the coupled systems remain most of the time in the principal LS configuration \((\tau = \tau_0)\) but occasionally visits other close LS configurations \((\tau = \tau_n, n = 1, 2, 3, ...)\) too.

So far, most of the studies\(^4,5,7\) on transition route from CS to PS in chaotic systems with coupling strength are theoretical. In Ref. 6, experimental observations on LS were presented for two nonidentical Rössler systems (electronic circuit version). In this work, we report our experimental results on the effect of coupling strength on synchronization of two nonidentical Chua’s oscillators. We have observed different regimes of synchronization by changing the coupling strength between the two oscillators. In order to verify the extent of synchronization between the driver and response circuits, a similarity function has been used here, which is introduced in Ref. 4 and later used by many.\(^6,7\) The similarity function between the scalar signals of the driver and the response is defined by

\[
S^2(t) = \frac{\langle (X_d(t - s) - X_r(t))^2 \rangle}{\langle (X_d(t))^2 \rangle^{1/2}} \langle (X_r(t))^2 \rangle^{1/2},
\]

where \(X_1 = [X_d, Y_d, Z_d]^T\) and \(X_2 = [X_r, Y_r, Z_r]^T\) are the state vectors of the driver and response oscillators, respectively. \(\tau\) is the lag time and \(T\) denotes transpose of the matrices. Effectively \(S(\tau)\) gives a measure of phase lag between the driver and response signals. The similarity function shows a global minimum \(\sigma_0 = S^2_{\text{min}}(\tau_0)\) with a lag minima \(\tau_{\text{min}} = \tau_0\), which indicates the existence of a principal lag time between the interacting systems. There also exist other local minima \(\sigma_n = S^2_{\text{min}}(\tau_n)\) for \(\tau_n \neq 0\ (n = 1, 2, 3, ...)\). In case of CS, \(X_d(t - \tau_0) = X_r(t)\) where \(\sigma_0 = 0\) for \(\tau_0 = 0\), whereas in case of LS, \(X_d(t - \tau_0) = X_r(t)\) is satisfied for global minimum \(\sigma_0 = 0\) and \(\tau_0 \neq 0\). Both PS and ILS regimes are characterized by non-

| TABLE I. Component values of driver and response oscillator circuits. |
|-----------------|-----------------|
| **Driver oscillator** | **Response oscillator** |
| \(L_1 = 20.3\ \text{mH}\), \(C_1 = 9.56\ \text{nF}\), \(C_2 = 95.9\ \text{nF}\) | \(L_2 = 20.51\ \text{mH}\), \(C_1 = 9.93\ \text{nF}\), \(C_2 = 93.3\ \text{nF}\) |
| \(R_{11} = 46.72\ \text{Ω}\) (variable) | \(R_{12} = 47.4\ \text{Ω}\) |
| \(R_2 = 2\ \text{kΩ}\) (variable) | \(R_2 = 2\ \text{kΩ}\) |
| \(R_3 = 3224\ \text{Ω}\), \(R_3 = 21.33\ \text{kΩ}\), \(R_4 = 21.3\ \text{Ω}\) | \(R_3 = 3194\ \text{Ω}\), \(R_8 = 21.33\ \text{kΩ}\), \(R_9 = 21.32\ \text{kΩ}\) |
| \(R_4 = 2153\ \text{Ω}\), \(R_5 = 220.6\ \text{Ω}\), \(R_6 = 221.6\ \text{Ω}\) | \(R_{10} = 2111\ \text{Ω}\), \(R_{11} = 216.2\ \text{Ω}\), \(R_{12} = 220\ \text{Ω}\) |
FIG. 2. Oscilloscope display: state space plot of response signal $X_r(t)$ along $X$ axis and driver signal $X_d(t)$ along $Y$ axis, (a) CS, $R_c = 51 \Omega$; (b) LS, $R_c = 718 \Omega$; (c) ILS, $R_c = 2256 \Omega$; (d) PS, $R_c = 4875 \Omega$. The driver is single scroll chaotic ($R_d = 1449 \Omega$) and the response is point attractor ($R_r = 2099 \Omega$).

FIG. 3. $X_d(t - \tau_0)$ vs $X_r(t)$ plots in the upper row shows (a) LS regime, $R_c = 718 \Omega$; (b) ILS regime, $R_c = 2256 \Omega$; (c) PS regime, $R_c = 4875 \Omega$. $X_d(t - \tau_0)$ vs $X_r(t)$ plots in lower row for (d) LS, $\tau_0 = 4 \mu s$; (e) ILS, $\tau_0 = 12 \mu s$; and (f) PS, $\tau_0 = 24 \mu s$. Large intermittent spikes bursts in (e) indicates ILS.
zero $\sigma_0$ with comparatively higher values of $\sigma_n$ ($\tau_n \neq 0$) reflecting the coexistence of many lag configurations. However, the nonzero $\sigma_0$ is appreciably small in the ILS regime in comparison to its large value in the PS regime.

II. EXPERIMENTAL CIRCUIT

We started with two nonidentical Chua’s oscillators, since it is difficult to design two identical oscillators. The circuit diagram of two coupled Chua’s oscillators is shown in Fig. 1. Each Chua’s oscillator consists of three energy storing elements as one inductor $L_{1,2}$ with a series resistance $R_{01,02}$, two capacitors $C_{1,3}, C_{2,4}$ and a nonlinear resistance, which is approximated by a piecewise linear function and simulated by using two Op-amps (uA741). The coupling is made unidirectional by using another Op-amp with a series resistance $R_c$, which decides the coupling strength. When the coupling resistance is increased, the coupling strength decreases and vice versa. The component values of the coupled oscillators are given in Table I. The circuit equations, in dimensionless form, are given by

\[
\frac{dX_d}{d\tau} = \alpha_d [Y_d - X_d - f(X_d)], \tag{2a}
\]

\[
\frac{dY_d}{d\tau} = X_d - Y_d + Z_d, \tag{2b}
\]

\[
\frac{dX_r}{d\tau} = \alpha_r [Y_r - X_r - f(X_r)] + \alpha_r \frac{R_c}{R_c} (X_d - X_r), \tag{2d}
\]

\[
\frac{dY_r}{d\tau} = X_r - Y_r + Z_r, \tag{2e}
\]

\[
\frac{dZ_r}{d\tau} = -\beta_r Y_r - \gamma_r Z_r, \tag{2f}
\]

and the piecewise linear functions $f(\cdot), g(\cdot)$ are given, in general form, by

\[
f(X) = bX + 0.5(a-b)(|X+1| - |X-1|), \tag{2g}
\]
FIG. 6. Oscilloscope display of state space plots of $X_r(t)$ along $X$ axis and $X_d(t)$ along $Y$ axis (a) CS, $R_c = 21 \, \Omega$; (b) LS, $R_c = 1550 \, \Omega$; (c) ILS, $R_c = 3231 \, \Omega$; (d) PS, $R_c = 7549 \, \Omega$. The driver is single scroll chaotic ($R_d = 1446 \, \Omega$) and the response is period-1 ($R_r = 1570 \, \Omega$).

FIG. 7. $X_d(t - \tau_0)$ vs $X_r(t)$ plots in upper row for (a) LS regime, $R_c = 1550 \, \Omega$; (b) ILS regime, $R_c = 3231 \, \Omega$; (c) PS regime, $R_c = 7549 \, \Omega$ and their difference $X_d(t - \tau_0) - X_r(t)$ plots with time in lower row for (d) $\tau_0 = 2 \, \mu s$ (LS), (e) $\tau_0 = 6 \, \mu s$ (ILS), and (f) $\tau_0 = 6 \, \mu s$ (PS). Large intermittent spikes bursts in (e) indicates ILS.
where \(a, b\) are the slopes of the inner region and outer regions, respectively, of the piecewise linear characteristics of the nonlinear resistor. The characteristics of the nonlinear resistors are the driving point characteristics of the Op-amps simulating the piecewise linear functions. The parameters are defined as

\[
a_d = \frac{C_2}{C_1}, \quad b_d = \frac{(R_d)^2 C_2}{L_1}, \quad \gamma_d = \frac{(R_d)}{(R_d)} \beta_d, \quad \alpha_d = \frac{C_4}{C_3}, \quad \beta_d = (R_d)^2 C_4/L_2, \quad \gamma_d = \frac{(R_d)}{(R_d)} \beta_d.
\]

Subscripts \(d\) and \(r\) denote the driver and response, respectively.

\[
X_d = V_{C1}, \quad Y_d = V_{C2}, \quad Z_d = R_d I_1 \quad \text{and} \quad X_r = V_{C3}, \quad Y_r = V_{C4}, \quad Z_r = R_r I_2
\]

are the state variables of the driver and response oscillators, respectively, where all \(V_C\) voltages are measured at respective capacitor nodes. \(R_d/R_r\) is the coupling parameter. \(\alpha_d\) and \(R_r\) are constant once the circuit components are fixed, the coupling resistance \(R_c\) decides the coupling strength.

**III. RESULTS**

In this paper, experimental results are presented on synchronization of two unidirectionally coupled Chua’s oscillators in drive–response mode. The resistors \(R_d\) and \(R_r\) in the driver and the response are taken as control parameters to obtain point attractor, single scroll, double scroll, and intermediate periodic states. Measurements of \(X, Y\) scalar variables as circuit node voltages of the coupled oscillators have been made and displayed through a digital oscilloscope (TEKTRONIX model TDS 220). For data acquisition, we used the TEKTRONIX software (WaveStar) available for the specific oscilloscope. The effect of coupling strength on the similarity function has been studied for different cases of oscillatory modes in the driver and the response oscillators. The similarity functions have been calculated from measured data. Five different cases have been investigated where the driver is either single scroll or double scroll chaotic. The response is in either of the states, namely, point attractor, period-1, single scroll chaotic. We decrease the coupling strength by increasing the coupling resistance when transitions from CS to PS have been observed in all cases, mainly.

![Phase difference plots with time](image1)

**FIG. 8.** Phase difference \(\phi_d(t) - \phi_r(t)\) plots with time in (a) CS regime \((R_c = 21 \Omega)\) (b) PS regime \((R_c = 7549 \Omega)\). The driver is single scroll chaotic and response is period-1.

![Similarity function plots](image2)

**FIG. 9.** Similarity function: the driver is single scroll chaotic and response is period-1. Different synchronization regimes are shown (a) CS \((R_c = 21 \Omega)\) in solid line, LS \((R_c = 1550 \Omega)\) in dotted line, ILS \((3231 \Omega)\) in dotted–dashed line; (b) global minima region is zoomed in the X-axis scale \(\tau = 30 \mu s\) for CS \((\alpha_d = 1.7 \times 10^{-3}, \tau_0 = 0)\), LS \((\alpha_d = 5 \times 10^{-3}, \tau_0 = 2 \mu s)\), ILS \((\alpha_d = 4 \times 10^{-2}, \tau_0 = 6 \mu s)\), and PS \((\alpha_d = 0.478, \tau_0 = 6 \mu s)\). Line trace notations are same as in (a). PS regime is not shown in (b) due to its much larger value of \(\tau_0\), which disturbs visual clarity of other synchronization regimes.
through two different routes for single scroll driver and corresponding response (point attractor, period-1 and single scroll chaotic). One route is through LS and ILS and the other route is through intermittency. But in case of double scroll driver and selected response oscillator (point attractor, period-1), both LS and ILS have been observed similar to single scroll cases, moreover, an intermediate intermittency regime is found to exist between ILS and PS but close to PS regime. LS and ILS are identified by the existence of $\sigma_n$ and $\tau_n$ ($n=0,1,2,3,...$), which are calculated by using the similarity function. We estimated the phases $\phi_{d,i}(t)$ of the driver and response by using Hilbert transform [11] on measured scalar signals to identify the PS regime.

Case I: Before coupling, the driver is single scroll chaotic and the response is in either of the following three states, (i) point attractor, (ii) period-1, (iii) single scroll chaotic.

In Case II(i), the driver is single scroll chaotic ($R_c=51\ \Omega$) and the response is kept at point attractor ($R_r=2099\ \Omega$). Transition from CS to PS through LS and ILS has been observed as shown in Fig. 2. For strong coupling ($R_c=51\ \Omega$), the synchronization manifold in Fig. 2(a) as $X_d(t)$ vs $X_r(t)$ plot is a $45^\circ$ line indicating CS. When the coupling resistance is successively increased to $R_c=718\ \Omega$, $2256\ \Omega$, and $4875\ \Omega$, consequent LS, ILS, and PS regimes are seen in Figs. 2(b)–2(d). $X_d(t-\tau)-X_r(t)$ plots and their differences $X_d(t-\tau)-X_r(t)$ in Figs. 3(a) and 3(b) and Figs. 3(d) and 3(e), respectively, are presented for related principal lag times $\tau_0=4\ \mu s$, $12\ \mu s$. The values of $\tau_0$ for different regimes are calculated by using the similarity function. The measured driver and response signals with a shifted time $\tau_0=4\ \mu s$ in Fig. 3(a) confirmed LS regime when the synchronization manifold is compressed to a $45^\circ$ line. The difference $X_d(t-\tau)-X_r(t)$ in Fig. 3(d), in this LS regime, shows regular spikes but within a small bound. With further decrease in coupling strength, the intermittent spike bursts are prominent in the ILS regime ($R_c=2256\ \Omega$), as seen in Figs. 3(b) and 3(e) [see also the oscilloscope display in Fig. 2(c)]. For $\tau_0=12\ \mu s$, the synchronization manifold in Fig. 3(b) is compressed to $45^\circ$ line yet it shows some bubbling [12] in the lower part, which is reflected as large intermittent spikes in Fig. 3(e) indicating intermittent nonsynchronous behavior. This distinguishes between LS and ILS regimes with mark difference in the height of spikes and in its intermittent behavior. We further increase in the coupling resistance to $R_c=4875\ \Omega$, PS regime is reached. The amplitudes of the driver and the response are uncorrelated here as shown in Figs. 3(c) and 3(f). The phase locking is confirmed by the phase difference $\phi_d(t)-\phi_r(t)$ in Fig. 4(b). The phase difference remains mostly bounded to zero in CS regimes [Fig. 4(a)] except rare isolated points of $2\pi$ phase jumps. These rare $2\pi$ phase jumps occur due to occasional zero crossing of the driver signal. Phase calculation [13] using Hilbert transform shows $2\pi$ jump, whenever the slope of the signal changes sign after zero crossing. In PS regime [Fig. 4(b)], the phase difference shows occasional $2\pi$ phase jumps otherwise bounded.

The similarity functions $S(\tau)$ are shown in Fig. 5 for different synchronization regimes. All $S(\tau)$ traces in Fig. 5(a) show periodic variations with a number of local minima $\sigma_n$ at regular intervals of lag time $\tau_0$ ($n=0,1,2,3,...$). But they have global minima $\sigma_0(\tau_0)$ at principal lag time $\tau_0$ (1). The similarity functions are zoomed in Fig. 5(b) for global minima only. As the coupling strength is increased, in the other sense, the coupling resistance is decreased, the values of both $\sigma_0$ and $\tau_0$ decrease, which agrees with earlier observations [7] in theoretical investigations on the coupled...
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FIG. 11. Phase difference $\phi_d(t) - \phi_r(t)$ plot with time in (a) intermittency ($R_c = 4.36 \Omega$) in upper trace shows sudden $-2\pi$ and $-4\pi$ phase slips. The lower trace plots $X_d(t) - X_r(t)$ with time showing intermittent bursts consistent with the timings of phase slips. (b) PS regime ($R_c = 33.79 \Omega$). The driver and response are both single scroll chaotic.

Rössler system. For strong coupling [$R_c = 51 \Omega$, solid line in Fig. 5(b)], the global minimum and the principal lag time are found as $\sigma_0 = 0.004$ and $\tau_0 = 0$, which indicates complete synchrony between the driver and the response. In the case of LS, for weaker coupling ($R_c = 718 \Omega$, dotted line), the values of global minimum and principal lag times are found as $\sigma_0 = 10^{-2}$ and $\tau_0 = 4 \mu s$, respectively, as shown in Fig. 5(b) [see also Fig. 2(b) and Fig. 3(a)]. In the ILS regime ($R_c = 2256 \Omega$, dashed line), they are calculated as $\sigma_0 = 0.028$ and a larger $\tau_0 = 12 \mu s$. The similarity function, in the PS regime, for even weaker coupling ($R_c = 4875 \Omega$, dotted–dashed line), gives much larger $\sigma_0 = 0.062$ and $\tau_0 = 24 \mu s$.

In case I(ii), the driver is single scroll chaotic ($R_d = 1446 \Omega$) and the response is period-1 ($R_r = 1570 \Omega$) when successive LS and ILS regimes have also been observed between CS to PS regimes similar to Case I(i) but with relatively much weaker coupling strength. The oscilloscope displays of $X_d(t)$ vs $X_r(t)$ plots are given in Fig. 6 for different coupling strengths. Regimes of CS ($R_c = 21 \Omega$), LS ($R_c = 1550 \Omega$), ILS ($R_c = 3231 \Omega$), and PS ($R_c = 7549 \Omega$) appear in Figs. 6(a)–6(d). $X_d(t-\tau) - X_r(t)$ plots and their differences $X_d(t-\tau) - X_r(t)$ in Figs. 7(a)–7(c) (upper row) and in Figs. 7(d)–7(f) (lower low), respectively, are plotted for $\tau_0 = 2 \mu s$, $6 \mu s$ in LS and ILS regimes and also for $\tau_0 = 6 \mu s$ in the PS regime. The projection of the measured driver and response voltages with a shifted time $\tau_0 = 2 \mu s$ in Fig. 7(a) confirmed LS regime when the synchronization manifold is compressed to a 45° line. In the case of ILS, intermittent spike bursts are prominent as seen in Figs. 7(b) and 7(e). The amplitudes of the driver and the response are completely uncorrelated in the PS regime ($R_c = 7549 \Omega$) as shown in Figs. 7(c) and 7(f). The phase difference $\phi_d(t) - \phi_r(t)$ in Fig. 8(b) confirms the PS regime, which is bounded but with occasional $2\pi$ phase jumps. In CS regime, it is bounded to zero as shown in Fig. 8(a) except few isolated points of $2\pi$ phase jumps, which may be explained as in case I(i). The similarity functions are plotted in Fig. 9 for CS (solid line, $R_c = 21 \Omega$), LS (dotted line, $R_c = 1550 \Omega$), ILS (dashed line, $R_c = 3231 \Omega$), and PS (dotted–dashed line, $R_c = 7549 \Omega$) regimes, respectively. In this case, all similarity functions for different coupling strength again show periodic local minima in Fig. 9(a). The values of global minima (principal lag times) are obtained as $\sigma_0 = 1.7 \times 10^{-3}$ ($\tau_0 = 0$) and $\sigma_0 = 5 \times 10^{-3}$ ($\tau_0 = 2 \mu s$), $\sigma_0 = 4 \times 10^{-2}$ ($\tau_0 = 6 \mu s$), and $\sigma_0 = 0.478$ ($6 \mu s$) in the CS, LS, ILS, and PS regimes, respectively, as shown in Fig. 9(b).

In case I(iii), both the driver and the response are single scroll chaotic for $R_d = 1439 \Omega$, $R_r = 1441 \Omega$ before coupling. Transitions from CS to PS through intermittency route has been observed as similar to the theoretical results reported in Ref. 9, no LS and ILS regimes are found. However, in this...
case, CS can be maintained even for much weaker coupling strength \(R_c = 669 \, \Omega\). Fig. 10(a) as compared to cases I(i)–I(ii) while PS regimes has been achieved at a very weak coupling \(R_c = 33.79 \, k\Omega\). Fig. 10(c)]. Intermittent large nonsynchronous bursts appears in an intermediate regime between CS and PS for \(R_c = 4.36 \, k\Omega\) as shown in Fig. 10(b). The phase difference \(\phi_d(t) - \phi_s(t)\) in the upper trace and \(X_d(t) - X_s(t)\) plot in the lower trace are shown in Fig. 11(a) for this intermittency region. It may be noted that the phase difference is bounded except during the occurrence of nonsynchronous spiking bursts when sudden \(2\pi\) and even \(4\pi\) phase jumps are found. The timings of intermittency are consistent with the timings of phase jumps. The phase difference in the PS regime Fig. 11(b)] is bounded except few \(2\pi\) phase slips. The similarity functions in CS (solid line, \(\sigma_0 = 0.0001, \tau_0 = 0\)) and intermittency (dotted line, \(\sigma_0 = 0.801, \tau_0 = 0\)) regimes in Fig. 12 show no lag configurations. In the PS regimes (dotted–dashed line), global minimum and principal lag time are given as \(\sigma_0 = 1.907, \tau_0 = 116 \, \mu s\). The largest value of global minimum \(\sigma_0 = 5.632\) (dashed line, \(\tau_0 = 0\)) for coupling resistance \(R_c = 7.4 \, k\Omega\) is a quasiperiodic window of the response oscillator between intermittency and PS regimes.

Case II: Without coupling, the driver is a double scroll chaotic attractor and the response is kept in either of the states (i) point attractor (ii) period-1.

In case II(i), the driver is double scroll chaotic and the response is at point attractor for \(R_d = 1416 \, \Omega\) and \(R_c = 1926 \, \Omega\). In CS regime \(R_c = 65 \, \Omega\), the synchronization manifold in Fig. 13(a) is a \(45^\circ\) line. For successive weaker couplings, LS \([R_c = 1.1 \, kohm, \) Fig. 13(b)], ILS \([R_c = 1.955 \, k\Omega, \) Fig. 13(c)], and PS \([R_c = 10.13 \, k\Omega, \) Fig. 13(d)] are observed. \(X_d(t-\tau)\) vs \(X_s(t)\) plots are shown in Figs. 14(a) and 14(b). In case of LS regime, it is compressed to thin lines [Fig. 14(a)] for principal lag time \(\tau_0 = 4 \, \mu s\), resembling the original piecewise linear characteristics of the nonlinear resistance of Chua’s oscillator. It may be noted that the piecewise linear characteristics of the nonlinear resistance in Chua’s oscillator has two slopes \(a\) and \(b\), where \(a\) is the slope of the inner region around the equilibrium at origin and \(b\) is the slopes of the outer regions around two mirror symmetric equilibria. In ILS regime, \(X_d(t-\tau)\) vs \(X_s(t)\) plot in Fig. 14(b) also shows thinning of the synchronization manifold for \(\tau_0 = 4 \, \mu s\) but there exists some width in the middle. Plots of the difference \(X_d(t-\tau) - X_s(t)\) with time are given in Figs. 14(c) and 14(d). A clear distinction between the LS and the ILS regimes is observed. The difference plot is not a horizontal line but it is as similar to the bi-symmetric time series of the double scroll. The delayed difference in Fig. 14(c) is bounded to small spikes in mirror symmetric positions confirming principal lag time \(\tau_0 = 4 \, \mu s\) in the LS regime while it shows intermittent spiking bursts in Fig. 14(d) for \(\tau_0 = 4 \, \mu s\) in the ILS regime. An interesting phenomenon of intermittent spike bursts has been observed near the PS regime after ILS, which is not observed in single scroll cases. In double scroll Chua’s oscillator, the system spends more time in either of the two symmetric lobes around their corresponding mirror symmetric equilibrium points and the trajectory switches between the two through the equilibrium point at the origin. Beyond the LS and ILS regimes, an inner loop starts growing as the coupling strength is weakened. The inner loop becomes large and large with two mirror symmetric lobes as shown in the oscilloscope display \((X_d vs X_s)\) plots in Fig. 15(a) for \(R_c = 6.8 \, k\Omega\). With increasing coupling resistance, the inner loop, ultimately, settles to a dense double scroll attractor around the equilibrium point at origin.
FIG. 14. $X_d(t - \tau_0)$ vs $X_r(t)$ plots in upper row for (a) LS regime, $R_c = 1.1\, k\Omega$; (b) ILS regime, $R_c = 1.955\, k\Omega$, and their difference $X_d(t - \tau_0) - X_r(t)$ plots with time in lower row for (c) LS, $\tau_0 = 4\, \mu s$; (d) ILS, $\tau_0 = 4\, \mu s$. Large intermittent spikes bursts in (d) indicates ILS.

FIG. 15. Oscilloscope display of state space plots of $X_r(t)$ along $X$ axis and $X_d(t)$ along $Y$ axis (a) $R_c = 6.8\, k\Omega$, (b) $R_c = 9.15\, k\Omega$, (c) $R_c = 9.6\, k\Omega$, (d) the difference $X_d(t) - X_r(t)$ plot with time of (b) for $R_c = 9.15\, k\Omega$. The driver is double scroll chaotic ($R_d = 1416\, \Omega$) and the response is point attractor ($R_r = 1926\, \Omega$).
as shown in Figs. 15(b) and 15(c). Some intermittent spiking bursts are clearly seen near PS regime for \(R_c = 9.15 \, k\Omega\), 9.6 \(k\Omega\). Spiking bursts are clear in the difference plot of \(X_r(t) - X_d(t)\) with time in Fig. 15(d) that corresponds to the situation shown in Fig. 15(b). With further weakening of the coupling strength \((R_c = 10.13 \, k\Omega)\), the intermittent bursts disappear to a symmetric double scroll as shown in Fig. 13(d). The phase differences \(\phi_d(t) - \phi_r(t)\) in Figs. 16(a) and 16(b) for CS \((R_c = 65 \, \Omega)\) and PS \((R_c = 10.13 \, k\Omega)\) remain mostly bounded, however, a few erratic \(2\pi\) jumps occur in the PS regime. Indeed, some smearing dots as \(\pi\)-phase jumps are also found, which are due to errors in phase calculation by Hilbert transform as it is dependent to DC bias. The phase calculation is sensitive\(^{11}\) to changes in slope of the signal near origin, which affects it in double scroll situation in particular. In double scroll, the signal always change sign during its switching between two mirror symmetric position across the origin. Error in phase calculation to the extent of \(2\pi\) jumps occurs during these changes in sign of the slope of signal.

The similarity functions in Fig. 17(a) show lag configurations for different coupling strength except in the CS regime \((R_c = 65 \, \Omega)\). In the CS regime, the global minimum and principal lag time are given as \(\sigma_0 = 0.002\) and \(\tau_0 = 0\), and as \(\sigma_0 = 0.0458\) and \(\tau_0 = 4 \mu s\) in the LS regime \((R_c = 1.1 \, k\Omega)\). In case of ILS, the global minima becomes larger \((\sigma_0 = 0.097)\) while the principal lag time remains same as \(\tau_0 = 4 \mu s\). Both global minima and principal lags are very large both during intermittency \((\sigma_0 = 0.45, \tau_0 = 48 \mu s)\) and in PS regime \((\sigma_0 = 1.09, \tau_0 = 36 \mu s)\).

In case II(ii), the driver is double scroll \((R_d = 1426 \, \Omega)\) again but the response is period-1 \((R_r = 1570 \, \Omega)\) before coupling. Successive transitions from CS \((R_c = 65 \, \Omega)\), LS \((R_c = 2.033 \, k\Omega)\), ILS \((R_c = 3 \, k\Omega)\), intermittency \((R_c = 9.6 \, k\Omega)\) and then to PS regime \((R_c = 48 \, k\Omega)\) have been observed (Fig. 18) for weaker and weaker coupling as similar as case II(ii). The regimes of LS and ILS are confirmed in \(X_d(t - \tau)\) vs \(X_r(t)\) plots with time in Figs. 19(a) and 19(b) and
FIG. 18. Oscilloscope display of state space plots of $X_r(t)$ along $X$ axis and $X_d(t)$ along $Y$ axis (a) CS, $R_c = 65 \, \text{k}\Omega$; (b) LS, $R_c = 2.033 \, \text{k}\Omega$; (c) ILS, $R_c = 3 \, \text{k}\Omega$; (d) PS, $R_c = 48 \, \text{k}\Omega$. The driver is double scroll chaotic ($R_d = 1426 \, \text{V}$) and the response is period-1 ($R_r = 1570 \, \text{V}$).

FIG. 19. $X_d(t - \tau_0)$ vs $X_r(t)$ plots in upper row for (a) LS regime, $R_c = 2.033 \, \text{k}\Omega$, (b) ILS regime, $R_c = 3 \, \text{k}\Omega$, and their difference $X_d(t - \tau_0) - X_r(t)$ plots with time in lower row for (c) LS, $\tau_0 = 2 \, \mu\text{s}$; (d) ILS, $\tau_0 = 8 \, \mu\text{s}$. Large intermittent spikes bursts in (d) indicates ILS.
their delayed difference $X_d(t-\tau) - X_r(t)$ plot with time in Figs. 19(c) and 19(d), by similar justifications as discussed in the previous case. The principal lag times used here as calculated by using similarity function shown in Figs. 20(a) and 20(b) are given by $\tau_0 = 2 \mu s$ ($\sigma_0 = 0.0215$) and $\tau_0 = 8 \mu s$ ($\sigma_0 = 0.108$) in LS and ILS, respectively. The intermittency is observed for $R_c = 9.6 \, k\Omega$ when intermittent bursts are seen in Figs. 21(a) and 21(b). The PS regime is reached at far weaker coupling $R_c = 48 \, k\Omega$. The interesting feature of the PS regime is that the system switches between two symmetric attractors as revealed in Fig. 18(d). We observed that $\pm 2\pi$ phase jumps occur during the transition from one attractor to another. The phase difference $\phi_d(t) - \phi_r(t)$ plot in Fig. 22(b) shows the occasional $\pm 2\pi$ phase jumps. Otherwise, the phase difference is bounded so long the system stays in either of the attractors. The phase difference in CS regime is bounded to zero [Fig. 22(a)]. Rare points of $2\pi$ jumps are found in the CS regime and indeed, some smearing dots as $\pi$-phase jumps are also found in PS regime. The explanation is as given above in case II(i). The global minima are given as $\sigma_0 = 0.409$ ($\tau_0 = 64 \mu s$) and $\sigma_0 = 1.9$ ($\tau_0 = 112 \mu s$) in the intermittency and the PS regimes, respectively.

**IV. DISCUSSION**

The basic intention of the authors is to find the transition routes from CS to PS as proposed by others in both theory and experiments on coupled chaotic systems. Two nonidentical Chua’s oscillators are coupled in unidirectional drive-response mode in our experiment. All similar components of the two oscillators have mismatch of about 1% except for resistors $R_d$ and $R_r$. The resistors $R_d$ and $R_r$ are taken as bifurcation parameters in the driver and response oscillators, respectively, to obtain various dynamical re-

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**FIG. 20.** Similarity function for different coupling resistances. The driver is double scroll chaotic and response is period-1. Different synchronization regimes are shown in (a) CS (65 \, \Omega) in solid line, LS (2.033 \, \Omega) in dotted line, ILS in dashed line (3 \, k\Omega), PS (48 \, \Omega) in bold-dotted line, and intermittency ($R_c = 9.6 \, k\Omega$) in dotted–dashed line, (b) global minima region is zoomed in the $X$ axis scale, $\tau = 150 \mu s$, line notations are the same as in (a). PS regime is not shown in (b).

**FIG. 21.** Oscilloscope display of state space plots of $X_r(t)$ along $X$ axis and $X_d(t)$ along $Y$ axis in (a) $R_c = 9.6 \, k\Omega$ and their difference $X_d(t) - X_r(t)$ with time in (b). The driver is double scroll chaotic ($R_d = 1426 \, \Omega$) and the response is period-1 ($R_r = 1570 \, \Omega$). Intermittent bursts are clear in (b).

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**FIG. 22.** Oscilloscope display of state space plots of $X_r(t)$ along $X$ axis and $X_d(t)$ along $Y$ axis in (a) $R_c = 9.6 \, k\Omega$ and their difference $X_d(t) - X_r(t)$ with time in (b). The driver is double scroll chaotic ($R_d = 1426 \, \Omega$) and the response is period-1 ($R_r = 1570 \, \Omega$). Intermittent bursts are clear in (b).
The existence of different lag configurations is identified from the values of \( t_s \) and \( \tau_0 \). In our experimental investigations, LS and ILS are observed in all cases except in case I(iii) when the driver and response are both chaotic. In case I(iii), transition from CS to PS regime occurs through intermittency route, no LS and ILS are observed. In double scroll situations, the transition from CS to PS is different from either of the routes noted above. Over and above the LS, ILS regimes, one can find intermittency region in both cases of point attractor and period-1 response. Our preliminary observations on such intermittency are reported here, which needs to be investigated further in the light of stability of synchronization manifold of coupled Chua’s oscillator as studied by others.\(^5,13\)

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12. “Bubbling term is used here in a literary sense, not in the strict sense of bubbling transition as explained in Ref. 13, which needs further rigorous proof.”