Chaos and Structures in a Chain of Mutually-Coupled Chua's Circuits

Grigory V. Osipov and Vladimir D. Shalfeev

Abstract— **The formation and interaction of structures in a chain of coupled Chua's circuits are investigated. Primary attention is focused on the control of spatio-temporal structures by choosing initial conditions and the values of coupling.**

I. Introduction

MANY PHYSICAL, biological and chemical processes are modelled by means of large ensembles of interacting bistable cells [1]–[3]. In the simplest case, each cell may be in one of two locally stable equilibrium states depending on initial conditions. The cell may be forced to move from one equilibrium state to the other (e.g., by a stepwise change of the instantaneous values of the variables describing the behavior of the cell), with the magnitude of forcing exceeding a certain critical level. When connected in a network, the cells having similar behavior may form spatial structures in which they are separated by transition regions. General regularities of structure formation have not been investigated in ample detail although there exists a broad literature on different problems concerning networks of coupled bistable cells which exibit transfer waves from one static state to another, and various phenomena concerning the interaction between transfer waves and evolution of the arbitrary initial perturbation, etc. [1], [3].

In this paper we consider the structure formation in a chain of coupled bistable cells of a much more complicated type, namely, Chua's circuits. Chua's circuit is known to be a self-oscillatory cell with complex dynamics the inherent features of which are the existence of two stable equilibrium states and periodic oscillations of different types, including chaotic oscillations [4], [5]. Our goal is to analyse the formation, interaction and transformation of spatial structures, both static and dynamic ones, i.e., we consider spatio-temporal structures.

Note that dynamical behavior, including the formation of one- and two-dimensional (1-D, 2-D) arrays of diffusively coupled Chua's circuits were analysed in [6]–[9] where quite a number of nontrivial spatio-temporal phenomena were revealed. In this paper, we consider a different type of coupling, namely, the coupling via coordinates (variables). The coupling of this type gives some advantages for theoretical analysis because it may be used as a control parameter. Following [10] we suppose that the neighboring cells are coupled linearly but, unlike [10], the coupling is taken to be mutual rather than unidirectional. Our primary attention is focused on the investigation of the regularities of structure formation and their control by choosing initial conditions in individual cells and the value of coupling between cells.

Manuscript received December 16, 1994; revised March 26, 1995. This work was supported in part by the Russian Foundation for Basic Research (Grant 93-02-15424) and by the International Association for the Promotion of Co-Operation With Scientists from the Independent States of the Former Soviet Union (Grant INTAS-94-2899). This paper was recommended by Guest Editor L. O. Chua.

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IEEE Log Number 9414463.

II. A Model of 1-D CNN of Chua's Circuits

The dynamics of a chain of coupled Chua's circuits may be described by a system of ordinary differential equations (ODE) [4], [5], [10]

$$\begin{aligned} \dot{x}_j &= \alpha\{y_j - h(x_j)\} + d_1 x_{j-1} + d_2 x_{j+1} \\ \dot{y}_j &= x_j - y_j + z_j \\ \dot{z}_j &= -\beta y_j, h(x_j) = c x_j - \frac{2c_0 x_j}{1 + c_0^2 x_j^2}, \\ & j = 1, 2, \cdots, N \end{aligned} \tag{1}$$

with boundary conditions

$$x_0(t) = x_{N+1}(t) \equiv 0. \tag{2}$$

The idealization and notation adopted in [10] are used in model (1). For simplicity we restrict ourselves to the case of identical coupling coefficients, i.e. we set $d_1 = d_2 \equiv d$.

Equation (1) is a model of a one-dimensional cellular nonlinear network (CNN) where Chua's circuits are used as cells [11], [12]. Partitioning of the plane of the parameters (α, β) into regions of different behavior in the absence of coupling between cells ($d = 0$) is presented in Fig. 1 for an individual Chua's circuit [10]. Partial dynamics of such an individual cell exhibits a rich choice of regimes ranging from static to chaotic ones. Therefore, it is natural to expect that a broad variety of static and dynamic structures may be formed in a coupled CNN of Chua's circuits. It is not our intention to give a complete classification of such structures. Instead, we will restrict our consideration to results of computer experiments on model (1) and attempt to understand the regularities of the formation of structures (clusters), their interaction and the potential controlling the processes of cluster formation by choosing initial conditions and coupling coefficients.

III. Static Clusters

We will refer to the groups formed by neighboring cells in qualitatively identical states as clusters and to the partioning of the CNN into different clusters as clusterization. With the characteristic features of an individual Chua's circuit taken

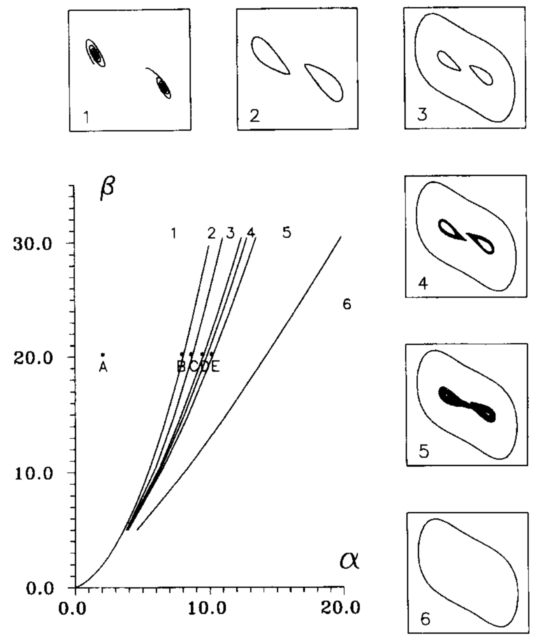


Fig. 1. Partitioning of the $\alpha - \beta$ parameter plane of Chua's circuit into regions of different regimes, as depicted by the insets corresponding to the parameter points A, B, C, D, E.

into account, the clusters for the cells with $x_j > 0$ at any t will be called "+" clusters and those for which $x_j < 0$ at any t, "−" clusters.

For a bistable medium in which each cell may be in one of two equilibrium states whose coordinates are not odd symmetric (i.e., symmetrical with respect to the origin), there may exist waves which transfer the cells from a "less" to a "more" stable state. As a consequence, a spatially homogeneous distribution must eventually be established throughout the medium. The transfer from one to the other stable state occurs as a result of external forcing. Both stable equilibrium states

$$O^-\left(x = -\sqrt{\frac{2c_0 - c_1}{c_0^2 c_1}}, y = 0, z = +\sqrt{\frac{2c_0 - c_1}{c_0^2 c_1}}\right)$$

and

$$O^+\left(x = +\sqrt{\frac{2c_0 - c_1}{c_0^2 c_1}}, y = 0, z = -\sqrt{\frac{2c_0 - c_1}{c_0^2 c_1}}\right)$$

of Chua's circuit are odd symmetric. Therefore, groups of cells in either of the two equilibrium states, i.e. "+" and "−" clusters, may coexist in the network (1). The control of such clusters is a very challenging problem. Such a problem may be formulated, for instance, by shifting the transition layer between clusters via appropriate perturbation, or by stabilization of the location of different clusters in space by choosing appropriate initial conditions and value of coupling.

Consider now clusterization. Let us first analyse a one-dimensional CNN of Chua's circuits each of which in the absence of coupling may be in either of the two equilibrium states O^- or O^+. We specify the values of the parameters of each cell deep in the domain corresponding to stable equlibrium states (point A in Fig. 1, $\alpha = 2, \beta = 20$), as well as a domain near the bifurcation curve for a transition to an oscillatory regime corresponding to an Andronov-Hopf bifurcation (point B in Fig. 1, $\alpha = 8, \beta = 20$). In all computer experiments we choose as initial conditions the following distribution of the variables (symmetric with respect to the middle of the chain):

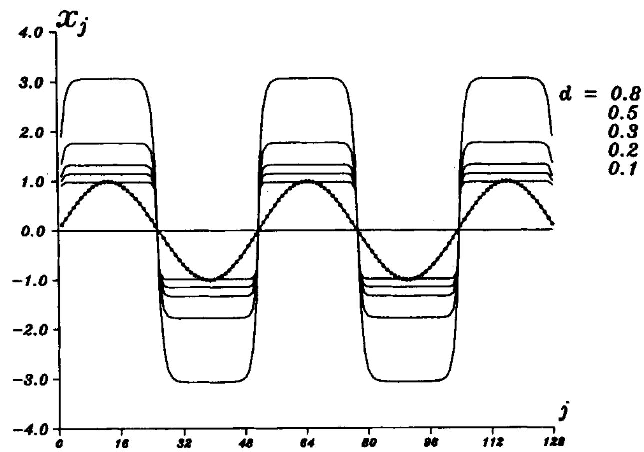


Fig. 2. Static clusters in a chain of Chua's circuits for different values of d. The initial distribution is identified by the dots.

$$\begin{aligned} x_j^0 &= \sin\left(\frac{2\pi j}{N} \cdot 1.5\right) \\ y_j^0 &= 0 \\ z_j^0 &= -x_j^0, j = 1, 2, \cdots, N. \end{aligned} \tag{3}$$

The coupling parameter d was varied from 0.01–0.8.

In the absence of coupling ($d = 0$), the image point of each cell in the three-dimensional phase space (x_j, y_j, z_j) for the chosen values of the parameters tends, as $t \to +\infty$, to the equilibrium state O^+ when $x_j^0 > 0, y_j^0 = 0, z_j^0 < 0$, and to the equilibrium state O^- when $x_j^0 < 0, y_j^0 = 0, z_j^0 > 0$. Observe that for $N = 128$, three static "+" and two static "−" clusters are formed in the chain of N uncoupled cells under (3). How does the clusterization process change when the cells are coupled? Numerical experiments carried out with $\alpha = 2$ and $\beta = 20$ showed that the boundaries between the "+" and "−" clusters remained unchanged for all values of d that we have choosen: the boundaries were determined unambiguously by initial conditions. Their position in the chain coincided for $d = 0$ and $d \neq 0$. Being odd symmetric, both the static distributions did not shift to the transition layer separating the "+" and "−" clusters. Fig. 2 shows the distributions of the variable $x_j = \overline{x}_j$ for several values of coupling coefficient d. The dots correspond to the initial x_j^0 distribution. As d is increased, the absolute magnitude of $\overline{x}_j$ grows for the cells inside each cluster. The quantities $\overline{x}_j, \overline{y}_j$ and $\overline{z}_j$ for the central cells of each clusters satisfy approximately the following relations

$$\begin{aligned} \overline{x}_{j-1} &= \overline{x}_j = \overline{x}_{j+1} \\ \overline{y}_{j-1} &= \overline{y}_j = \overline{y}_{j+1} \\ \overline{z}_{j-1} &= \overline{z}_j = \overline{z}_{j+1}. \end{aligned} \tag{4}$$

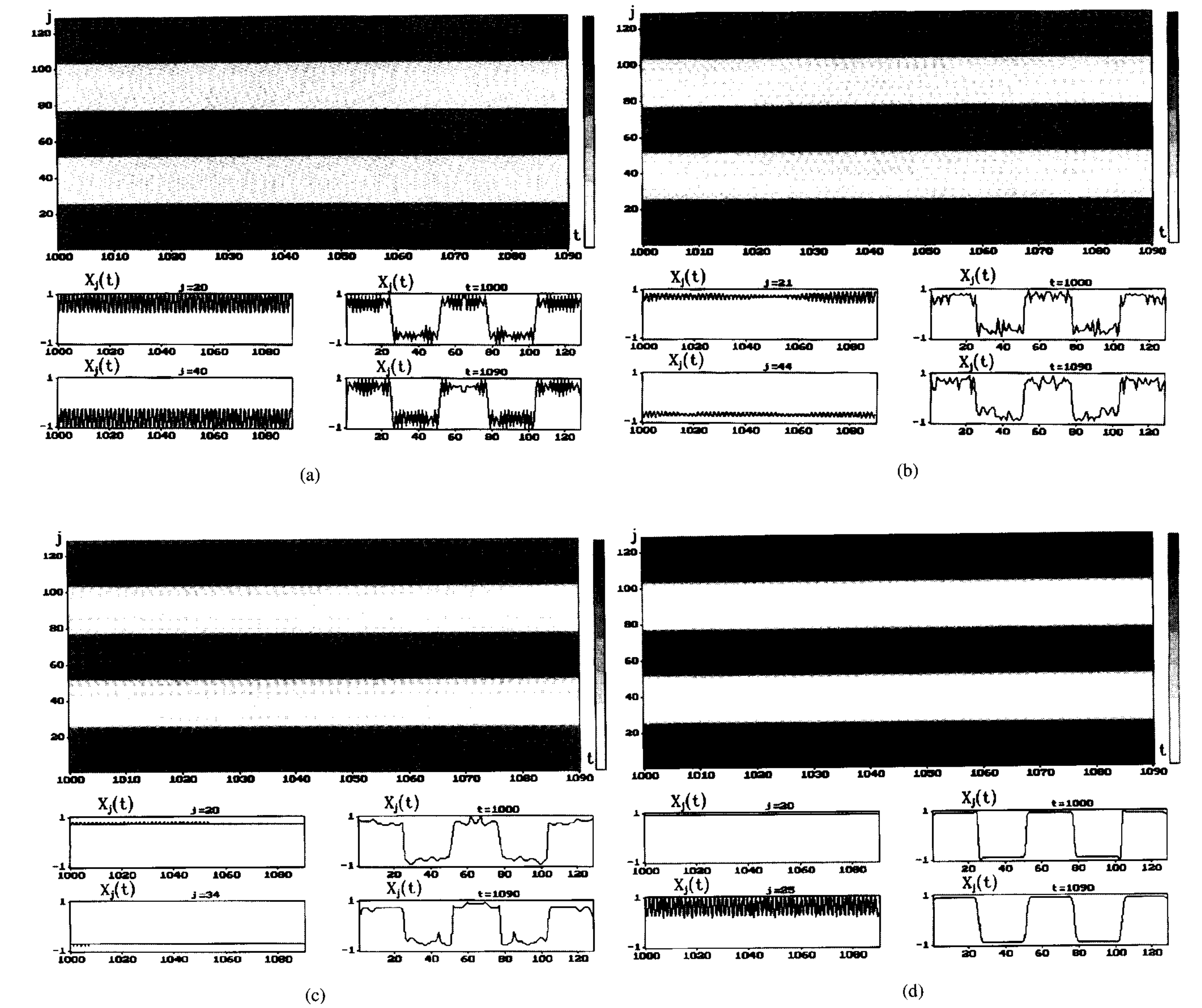


Fig. 3. (a) Space–time diagram of the variation of $x_j(t)$: $\alpha = 8.5, \beta = 20$, and $d = 0.1$. (b) Space–time diagram of the variation of $x_j(t)$: $\alpha = 8.5, \beta = 20$, and $d = 0.3$. (c) Space–time diagram of the variation of $x_j(t)$: $\alpha = 8.5, \beta = 20$, and $d = 0.5$. (d) Space–time diagram of the variation of $x_j(t)$: $\alpha = 8.5, \beta = 20$, and $d = 0.8$.

Therefore, by using (1) $\overline{x}_j, \overline{y}_j, \overline{z}_j$ can be derived from the system of equations

$$\begin{aligned} -\alpha h(\overline{x}_j) + 2d\overline{x}_j &= 0 \\ \overline{y}_j &= 0 \\ \overline{z}_j &= -\overline{x}_j \\ h(\overline{x}_j) &= c\overline{x}_j - \frac{2c_0\overline{x}_j}{1 + c_0^2\overline{x}_j^2} \end{aligned} \tag{5}$$

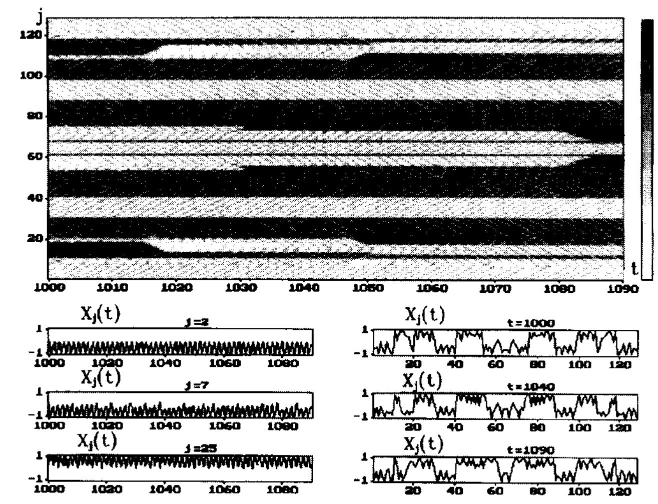
to give

$$\overline{x}_j^2 = \frac{2c_0 - (c_1 + 2d)}{(c_1 - 2d)c_0^2}.$$

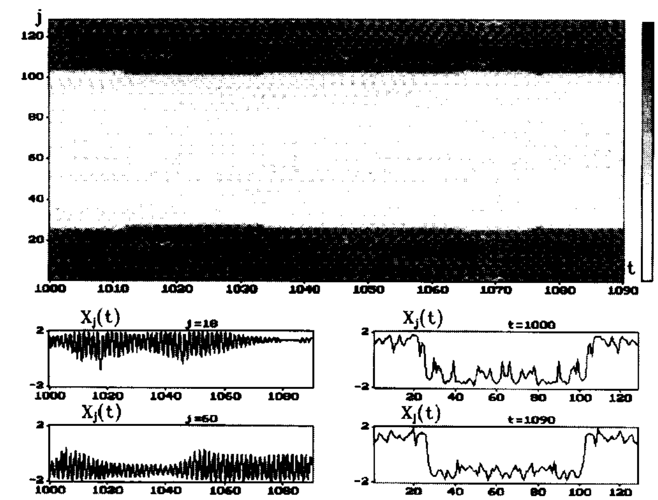
Our experiments with $\alpha = 8$ and $\beta = 20$, i.e. near the bifurcation curve corresponding to the Andronov-Hopf bifurcation (point B in Fig. 1), show that the "+" and "−" clusters do not change their position as d is increased, similar to the previous case ($\alpha = 2, \beta = 20$). In this case, however, the effect of the boundary conditions and closeness of the parameters α and β to the domain of periodic regimes (domain 2 in Fig. 1) for sufficiently large d give rise to oscillatory motions at the boundaries of the clusters.

IV. Cluster Formation from Regularly Oscillating Cells

In another series of experiments we choose the values of the parameters (α and β) so that each uncoupled cell operates in domain 2 of Fig. 1. This means that a regime of regular periodic oscillations is realized in each cell in either the halfspace $x_j < 0, z_j > 0$ or in $x_j > 0, z_j < 0$, depending on initial conditions. The space–time diagrams of the variation of $x_j(t)$ are given in Figs. 3–6. Here, the time t is laid off along the horizontal axis, and the number j of the cell in the chain along the vertical scale. The black color corresponds to the maximal and white to the minimal amplitude of $x_j(t)$. Examples of typical oscillograms for some cells $x(t)$ at $j =$ const and distributions $x(j)$ in the chain at some fixed instants of time ($t =$const) are also presented in Figs. 3–6.



(a)



(b)

Fig. 4. (a) Space–time diagram of the variation of $x_j(t)$: $\alpha = 9.4, \beta = 20$, and $d = 0.2$. (b) Space–time diagram of the variation of $x_j(t)$: $\alpha = 9.4, \beta = 20$, and $d = 0.8$.

Consider first the behavior of the chain for the parameters α and β specified at point C ($\alpha = 8.5, \beta = 20$), i.e., far from the bifurcation curve corresponding to the transition to chaotic oscillations (see Fig. 1). Computer experiments show that the regime of synchronized periodic oscillations in the cells does not break when a weak positive coupling ($d > 0$) is introduced, but the oscillations in the neighboring cells (except those near the transition boundaries) are almost out-of-phase (Fig. 3(a)). According to the initial distribution (3), three "+" and two "−" clusters consisting of periodically oscillating cells are formed. For $d = 0.3$ (Fig. 3(b)), the time evolution no longer remains regular throughout the chain. In this case, some cells persist to generate periodic oscillations, while in most of the other cells, a regime of chaotic oscillations analogous to a spiral Chua's attractor is realized. For $d = 0.5$, there appears a tendency for an almost in-phase oscillations in the neighboring cells (Fig. 3(c)). In this case, the intensity of oscillations is decreased markedly. For $d = 0.8$, the oscillations are quenched in most of the cells in the clusters, i.e. a static regime is established (Fig. 3(d)). In this case, only the boundary cells (e.g., $j = 25$) of the clusters (except the boundary cells of the chain) still oscillate periodically. With a further increase in d, a static regime is established in all cells of the chain.

The above observed phenomena may be explained by analysing the dynamics of some neighboring cells in the chain. It is readily understood that the coupling between cells may lead to changes both in the coordinates of the equilibrium states in the cells of the chain, as well as in the quantitative and qualitative characteristics of motions of the cells. Suppose, for example, that a regime is realized for which the values of the variables x_j coincide for some neighboring cells by virtue of initial conditions. Then each of these cells may be described by the system of equations

$$\begin{aligned} \dot{x}_j &= \alpha(y_j - h(x_j)) + 2dx_j \\ \dot{y}_j &= x_j - y_j + z_j \\ \dot{z}_j &= -\beta y_j. \end{aligned}$$

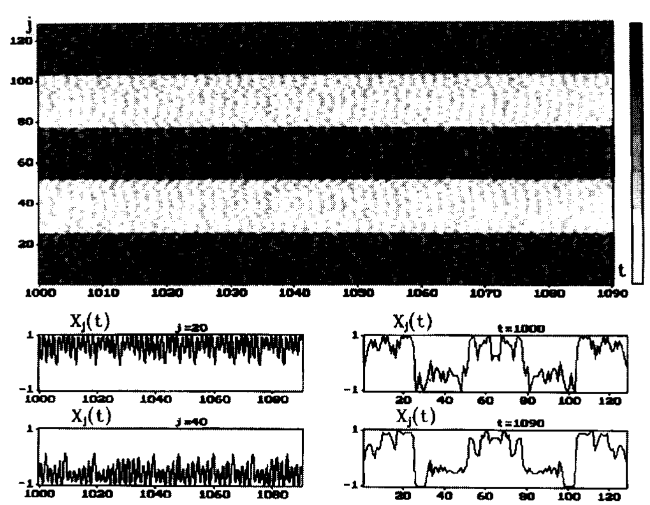
In this case, for fixed values of the parameters c_0, c_1, and d, the partitioning of the plane of the parameters (α, β) into regions corresponding to different attractors in the phase space are not identical to the corresponding partitioning for $d = 0$. Consequently, the coupling $d \neq 0$ may give rise to qualitative changes, compared to the case $d = 0$, in the type of motions which are realized in each cell of the chain.

Besides it is worthy of notice that in a long chain where broad "+" and "−" clusters are realized, the dynamics of the cells located near the boundaries between the clusters must be close to the dynamics of uncoupled cells because the condition $x_{j+1} \approx -x_{j-1}$ must be fulfilled. Experiments on long chains where the effect on the dynamics of boundary conditions is insignificant show that the cells located in the transition region between the clusters retain their individual (uncoupled) dynamics over a broad range of variation of d (Fig. 3(d)).

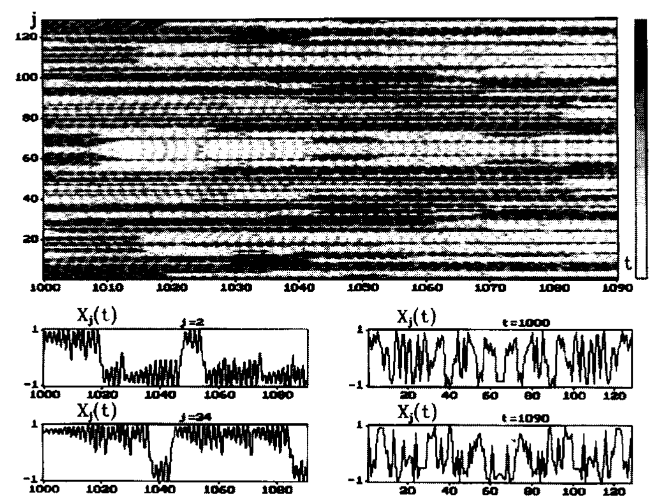
Consider now the parameters α and β for which a cascade of period doubling bifurcations of periodic motions (point D in Fig. 1, $d = 9.4, \beta = 20$) is initiated in an uncoupled Chua's circuit. We found that in this case the location of the "+" and "−" clusters is *not* determined unambiguously by the initial conditions, instead it depends strongly on the value of d and is not fixed in space (Fig. 4(a) and (b)). Evidently, this is explained by the fact that the introduced coupling is so significant that it leads to a global mixing in the chain and may change a regime of periodic oscillations, to a regime of chaotic oscillations having the form of a double-scroll Chua's attractor rather than a spiral attractor.

V. Cluster Formation of Chaotically Oscillating Cells

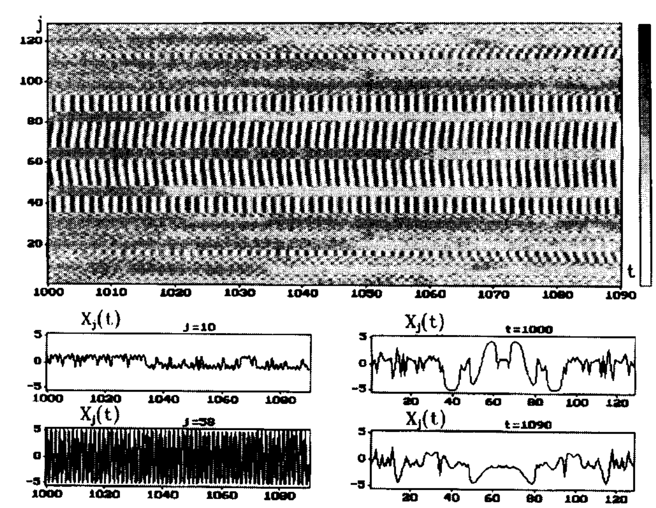
Let us now consider clusterization in the presence of chaotic oscillations in uncoupled cells. Computer experiments were carried out for $\alpha = 10.1$ and $\beta = 20$ (point E in Fig. 1). A chaotic regime the image of which in partial phase space may be one of two spiral Chua's attractors is realized in an uncoupled Chua's circuit for these values of the parameters. In this case, the initial conditions (3) determine the domains



(a)



(b)



(c)

Fig. 5. (a) Space–time diagram of the variation of $x_j(t)$: $\alpha = 10.1, \beta = 20$, and $d = 0.01$. (b) Space–time diagram of the variation of $x_j(t)$: $\alpha = 10.1, \beta = 20$, and $d = 0.05$. (c) Space–time diagram of the variation of $x_j(t)$: $\alpha = 10.1, \beta = 20$, and $d = 0.8$.

of the existence of the "+" and "−" clusters (Fig. 5(a)) only if the coupling is very weak, i.e., $d \leq 0.02$. As d increases, the oscillation intensity and the extent to which the cells affect each other increases, which leads to a mixing of oscillations in the chain and to the onset for each cell of another regime of chaotic oscillations corresponding to double-scroll (Fig. 5(b)). At the same time, a regime of periodic oscillations corresponding to a large limit cycle (enclosing both equilibrium states O^- and O^+) in an uncoupled Chua's circuit may be established for large d in some cells of the chain (Fig. 5(c)), e.g., at $j = 58$, because for $\beta = 20$ the interval for the parameter α in which chaotic regimes are realized is very small. Thus, in this case, both chaotic and regular oscillations coexist in the chain (Fig. 5(c)).

VI. Clusterization at Negative Coupling Coefficients

The situation is drastically different when the sign of the coupling coefficients $(d < 0)$ between the cells in a one-dimensional CNN of Chua's circuits is changed. Results of computer experiments for a chain with the parameters $\alpha = 8.5$ and $\beta = 20$ providing a regime of periodic oscillations for each uncoupled cell are given in Fig. 6. In-phase oscillations in the neighboring cells prevail for small d (Fig. 6(a)), while a regime of static distribution $\overline{x}_j$ having different signs in the neighboring cells is realized for large d (Fig. 6(e)). It is interesting to follow the appearance and onset of antiphase oscillations in the neighboring cells as we increase d. Individual small regions of antiphase oscillations appear first (Fig. 6(b)). These regions are nonstationary and the oscillations may be either regular or chaotic. With a further increase in d (Fig. 6(c)), there emerge stationary regions of antiphase oscillations which gradually spread through out the chain (Fig. 6(d),(e)).

An analogous mechanism for the transition to an antiphase distribution of variables along the chain is observed for the parameters α and β such that an uncoupled cell is in an equilibrium, or a chaotic regime.

VII. Conclusion

Numerical simulations of the nonlinear dynamics of a one-dimensional CNN of Chua's circuits enables us to summarize the following conclusions:

i) A rather broad variety of static and dynamic structures may be formed in the chain considered. The formation of structures depends significantly on initial conditions in the cells, boundary conditions in the chain, the type and value of coupling between the cells, and on the choice of the parameters of individual cells.

ii) For the coupling coefficients and boundary conditions considered in this paper, one can effectively control the structure formation process in the chain by choosing appropriate initial conditions in the individual cells.

It is to be expected that the phenomena of structure formation [13] have a rather general nature and may be extended to 2-D CNN, as well as to CNN with nonlinear couplings, etc.

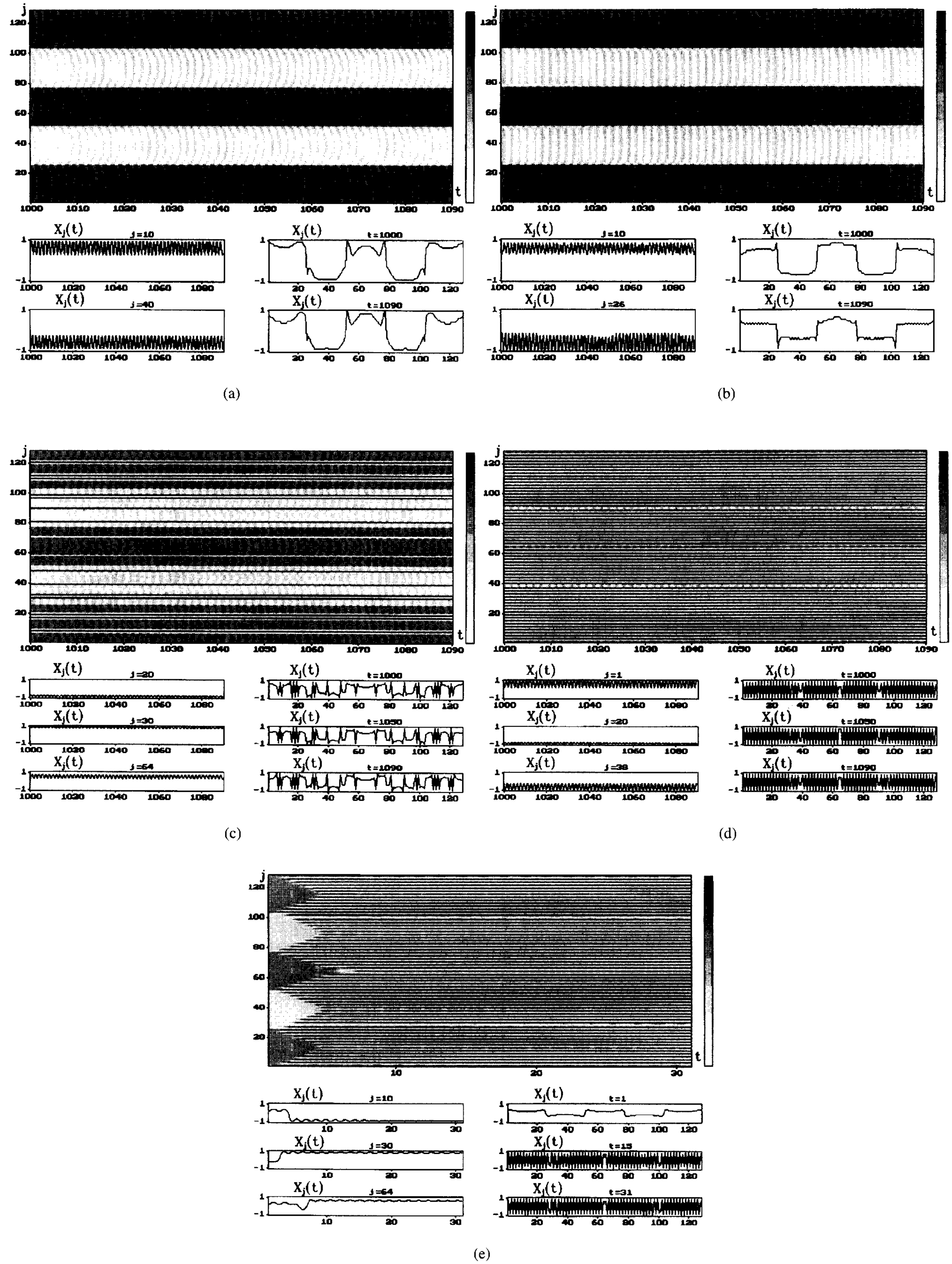


Fig. 6. (a) Space–time diagram of the variation of $x_j(t)$: $\alpha = 8.5, \beta = 20$, and $d = -0.1$. (b) Space–time diagram of the variation of $x_j(t)$: $\alpha = 8.5, \beta = 20$, and $d = -0.4$. (c) Space–time diagram of the variation of $x_j(t)$: $\alpha = 8.5, \beta = 20$, and $d = -0.5$. (d) Space–time diagram of the variation of $x_j(t)$: $\alpha = 8.5, \beta = 20$, and $d = -0.6$. (e) Space–time diagram of the variation of $x_j(t)$: $\alpha = 8.5, \beta = 20$, and $d = -1.0$.

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Grigory V. Osipov, for a photograph and biography, see this issue, p. 692.

Vladimir D. Shalfeev, for a photograph and biography, see this issue, p. 692.