International Journal of Bifurcation and Chaos, Vol. 9, No. 1 (1999) 295–303


# HOPF BIFURCATIONS AND DEGENERACIES IN CHUA'S CIRCUIT — A PERSPECTIVE FROM A FREQUENCY DOMAIN APPROACH

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Received March 1, 1998; Revised May 1, 1998

In the present work explicit formulas for analyzing the birth of limit cycles arising in the Chua's circuit through a Hopf bifurcation is provided. A local amplitude equation is derived using a frequency domain approach and harmonic balance approximations. Furthermore, the first Lyapunov index used to detect degenerate Hopf bifurcations is derived in terms of the parameters of the nonlinear circuit. A perspective for analyzing other bifurcations using this frequency domain approach is discussed.

## 1. Introduction

Chua's circuit has become a paradigm for complex oscillatory dynamics and chaos arising in simple electronic nonlinear circuits [Madan, 1993; Chua, 1994; Shil'nikov, 1994]. The bulk of papers regarding the complex dynamics in this circuit has focused on chaotic attractors, period-doubling bifurcations, period-adding bifurcations, and so on. In this paper, on the contrary, a study of Hopf bifurcation is performed using a frequency domain approach in a way reminiscent of the classical describing function method. The procedure consists of applying the harmonic balance method to provide the (local) amplitude solution for the emerging limit cycles. Moreover, the first Lyapunov index or curvature coefficient is obtained in terms of the relevant parameters of the circuit. The vanishing of the curvature coefficients allows us to study the birth of multiple periodic solutions in the unfoldings of the so-called degenerate Hopf bifurcation [Golubitsky & Langford, 1981].

This work follows the lines initiated in [Altman, 1993] regarding the dynamics of Hopf bifurcation in Chua's circuit. However, the procedure used here for approximating the amplitude of limit cycles does not use the center manifold theory or coordinate transformations. A frequency domain approach, closely related to control theory and the harmonic balance method are used to obtain the main results [Mees & Chua, 1979].

The motivation of this study comes from recent results obtained in [Khibnik *et al.*, 1993b] concerning the existence of multiple limit cycles around the symmetrical equilibria of Chua's circuit, as well as large amplitude cycles surrounding the three singular points. As we have used a smooth (third-order polynomial) nonlinearity to approximate the piecewise linear characteristics, this work shares similarities with continuous efforts made by other researchers in finding the maximum number

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of limit cycles in planar cubic systems [Lloyd *et al.*, 1988; Lloyd & Pearson, 1990; Zoladek, 1995]. Since Chua's equations consists of three first-order ODEs, its dynamics are more complicated than those arising from two first-order ODEs (also called planar systems). In particular, it is a challenge to plot the successive curvature coefficients to determine regions of multiple periodic solutions in the parameter space. This letter is the first attempt in this direction. Moreover, a discussion is presented concerning the extension of this method to detect other bifurcations using a higher-order harmonic balance.

## 2. Main Results

The smooth model of Chua's circuit [Altman, 1993; Khibnik *et al.*, 1993b] is given by

$$\begin{aligned}\dot{x} &= \alpha[y - \varphi(x)]\,,\\ \dot{y} &= x - y + z\,,\\ \dot{z} &= -\beta y\,,\end{aligned} \tag{1}$$

where $\varphi(x) = c_1x^3 + c_3x$, is a cubic polynomial nonlinearity, $\alpha$, $\beta$, $c_1$, and $c_3$ are system parameters. The parameters $\alpha$ and $\beta$ will be used as the main bifurcation parameters in the following, for the sake of clarity. After making the following change of coordinates to simplify the structure of the linear part

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

system (1) can be recast as follow:

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{2}\alpha c_3 x_1 + \alpha x_3 - \frac{1}{2}\alpha c_3 x_1 - \alpha c_1 x_1^3\,,\\ \dot{x}_2 &= \beta x_3\,,\\ \dot{x}_3 &= x_1 - x_2 - x_3\,.\end{aligned} \tag{2}$$

Choosing the following equivalent representation of Eq. (2) (see [Moiola & Chen, 1996] for more details about the methodology):

$$\begin{aligned}\dot{x} &= A(\alpha,\, \beta)x + Bg(Cx;\, \alpha,\, \beta)\,,\\ y &= C(\alpha,\, \beta)x\,,\end{aligned}$$

where

$$A = \begin{bmatrix} -\alpha c_3/2 & 0 & \alpha \\ 0 & 0 & \beta \\ 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C = [1 \quad 0 \quad 0]\,,$$

$$g(x;\, \alpha,\, \beta) = -\alpha\left(\frac{1}{2}c_3x_1 + c_1x_1^3\right),$$

we end up with a simple form for the linear transfer function $G(s;\, \alpha,\, \beta) = C[sI - A]^{-1}B$, where "$s$" is the variable of the Laplace transform,

$$\begin{aligned}&G(s;\,.\,,.)\\ &= \frac{s^2 + s + \beta}{s^3 + s^2\left(1 + \frac{1}{2}\alpha c_3\right) + s\left(\beta + \frac{1}{2}\alpha c_3 - \alpha\right) + \frac{1}{2}\alpha\beta c_3}\\ &= \frac{N(s)}{D(s)}\,.\end{aligned}$$

The Jacobian matrix to be used in the frequency domain formulation is

$$J = \frac{1}{2}\alpha c_3 + 3\alpha c_1 \hat{e}_1^2\,,$$

where $x_1 = -e_1$, $g(x_1) := f(e_1)$ and $\hat{e}_1$ is the equilibrium point obtained from

$$G(0;\,.\,,.)f(e_1) = \frac{2\beta}{\alpha\beta c_3}\left(\frac{1}{2}\alpha c_3 e_1 + \alpha c_1 e_1^3\right) = -e_1\,,$$

$$\Rightarrow \hat{e}_1^{(1)} = 0, \quad \text{and} \quad \hat{e}_1^{(2,\,3)} = \hat{e}_1^{(\pm)} = \pm\sqrt{\frac{-c_3}{c_1}}\,.$$

Notice that our selection of a unique representative variable $e_1$ simplifies the computation of the Hopf bifurcation formulas given in [Moiola & Chen, 1996] since the eigenvectors $v$ and $w^T$ are both equal to 1. However, explicit expressions for the original three variables can also be obtained, and in this case the expressions of the eigenvectors are slightly more complicated.

The following eigenlocus $G(s)J$ is then calculated about the symmetrical equilibrium points $\hat{e}_1^{(\pm)} = \pm\sqrt{-c_3/c_1}$ (called $P^{\pm}$ in the literature on Chua's circuit[1]) which give birth to a Hopf bifurcation under appropriate values of the system parameters:

$$\hat{\lambda} = G(s)J = \frac{-\frac{5}{2}\alpha c_3 N(s)}{D(s)}\,. \tag{3}$$

[1]Since $x_1 = -e_1$, then $P^+ = \hat{x}_1^{(+)} = -\sqrt{-c_3/c_1} = \hat{e}_1^{(-)} = -\hat{e}_1^{(+)}$

The Hopf bifurcation condition is obtained from Eq. (3) when $\hat{\lambda} = -1$ and $s = i\omega_0$, $\omega_0 \neq 0$ giving the following pair of equations

$$\omega_0^2 = \beta - 2\alpha c_3 - \alpha = \frac{-2\alpha c_3 \beta}{1 - 2\alpha c_3}\,. \tag{4}$$

Thus, a simple expression of the starting frequency of oscillations through the Hopf bifurcation mechanism at the symmetrical equilibrium points can be easily obtained from Eq. (4) as follows

$$\omega_0^2 = -2\alpha^2 c_3 (1 + 2c_3)\,. \tag{5}$$

Notice that to change (control) the frequency of emerging limit cycles the relevant parameters are $\alpha$ and $c_3$.

The following closed-loop transfer function is useful in the computation of the Hopf bifurcation formulas:

$$\begin{aligned} H(s;\, \alpha,\, \beta) &= \frac{G(s;\, \alpha,\, \beta)}{[1 + G(s;\, \alpha,\, \beta) J]} \\ &= \frac{N(s)}{N(s)(s - 2\alpha c_3) - \alpha s}\,. \end{aligned}$$

The Hopf bifurcation formulas needed for the computation of the amplitude equation and the first Lyapunov index are as follows:

$$\begin{aligned} V_{02} &= -\frac{1}{4} H(0; ., .) D_2 v \otimes \overline{v}\,, \\ &= \frac{3}{4\hat{e}_1^{(\pm)}}\,; \end{aligned} \tag{6}$$

$$\begin{aligned} V_{22}(\omega) &= -\frac{1}{4} H(i2\omega; ., .) D_2 v \otimes v\,, \\ &= -\frac{6\alpha c_1 \hat{e}_1^{(\pm)}}{4} H(i2\omega); \end{aligned} \tag{7}$$

$$\begin{aligned} p1(\omega) &= D_2 \left[\frac{1}{2}\overline{v} \otimes V_{22} + v \otimes V_{02}\right] \\ &\quad + \frac{1}{8} D_3 v \otimes v \otimes \overline{v}\,, \\ &= 6\alpha c_1 \left[\frac{1}{2} e_1^{(\pm)} V_{22}(\omega) + \frac{7}{8}\right], \end{aligned} \tag{8}$$

where $D_2$ and $D_3$ are the second and third order partial derivatives of $f(e_1)$ evaluated at the equilibrium point $\hat{e}_1^{(\pm)}$, $\otimes$ is the tensorial product (in this case, the scalar product), and the bar denotes the complex conjugate operation. The amplitude equation is obtained from the following approximation

$$\hat{\lambda} = -1 + \theta^2 \xi_1 = -1 - \theta^2 G(i\omega) p_1(\omega)\,, \tag{9}$$

where $\theta$ is a measure of the amplitude of the first harmonic and $\xi_1$ is a complex number. Then, from Eq. (9) a measure of the amplitudes of the emerging limit cycles from Hopf bifurcations can be written as follows:

$$\theta = \sqrt{-\frac{\hat{\lambda} + 1}{G(i\omega) p_1(\omega)}}\,. \tag{10}$$

The computation of the first Lyapunov index requires the evaluation of Eqs. (6)–(8) at criticality [$s = i\omega_0$, $\omega_0 \neq 0$ given by Eq. (4)]. Several simplifications yield the following expressions:

$$\begin{aligned} V_{02} &= -\frac{1}{4} H(0; ., .) D_2 v \otimes \overline{v}\,, \\ &= \frac{3}{4\hat{e}_1^{(\pm)}}\,, \end{aligned} \tag{11}$$

$$\begin{aligned} V_{22}(\omega_0) &= -\frac{1}{4} H(i2\omega_0; ., .) D_2 v \otimes v\,, \\ &= \frac{3}{4\omega_0 \hat{e}_1^{(\pm)}} \left\{\frac{-(1 + 6\alpha c_3)\omega_0 + i4\alpha c_3}{3(1 - 2\alpha c_3) + i6\omega_0}\right\}, \\ &= \frac{3}{4\omega_0 \hat{e}_1^{(\pm)}} v_2(\omega_0)\,, \end{aligned} \tag{12}$$

$$\begin{aligned} p_1(\omega_0) &= D_2 \left[\frac{1}{2}\overline{v} \otimes V_{22} + v \otimes V_{02}\right] + \frac{1}{8} D_3 v \otimes v \otimes \overline{v}\,, \\ &= 6\alpha c_1 \left[\frac{3}{8}\frac{v_2(\omega_0)}{\omega_0} + \frac{7}{8}\right]. \end{aligned} \tag{13}$$

Finally, the *curvature coefficient* (or the first Lyapunov index) can be expressed by

$$\begin{aligned} \sigma_1(\omega_0) &= -\Re\left\{\frac{w^T G(i\omega_0) p_1(\omega_0)}{w^T G' J v}\right\} \\ &= -\Re\left\{\frac{[\alpha(1 + 2\alpha c_3) + i\omega_0] p_1(\omega_0)}{-2\omega_0^2 - \frac{5}{2}\alpha c_3 + i2\omega_0 \left(1 - \frac{9}{2}\alpha c_3\right)}\right\}, \end{aligned} \tag{14}$$

where $\Re\{.\}$ denotes the real part of a complex number. Equating expression (14) to zero gives the degenerate Hopf bifurcations points whose unfoldings contain multiple limit cycles. This degenerate Hopf bifurcation curve can be continued in the parameter space $\alpha - \beta - c_3$ to search for other more complicated singularities as organizing centers of the dynamics. For this parameter set: $\alpha = 1.106691504$, $\beta = 1.09950956$, $c_1 = 1$ and $c_3 = -0.06934372403$ there is a Hopf bifurcation in the symmetrical equilibria $P^{\pm}$ with a frequency $\omega_0 = 0.3824948062$ having the first and second Lyapunov indexes equal to zero. According to the theory of degenerate Hopf bifurcations a structure of three nested limit cycles can be encountered for a suitable perturbation in the parameters $\alpha - \beta - c_3$.

Some simulations using LOCBIF [Khibnik *et al.*, 1993a] are presented below for illustration of the main results near the above mentioned singularity by fixing $c_1 = 1$ in all the cases considered. In Fig. 1, a stable limit cycle encircling $P^-$ is shown for $\alpha = 1.106691$, $\beta = 1.0991$, and $c_3 = -0.06934372$. The limit cycle is separated from the origin in the phase-plane (the left-top corner in Fig. 1).

Varying appropriately the parameters $\alpha$ and $\beta$ the stable limit cycle is deformed such that one of its extremes is near the origin (close to a saddle-loop separatrix bifurcation). This situation is depicted in Fig. 2 for $\alpha = 1.22043$, $\beta = 1.22$, and $c_3 = -0.06934372$.

Figures 3–5 show one large amplitude stable limit cycle surrounding the three equilibria and two unstable limit cycles surrounding $P^+$ and $P^-$. The simulations were obtained using different initial conditions in order to give an idea of the basins of attractions of the stable solutions.

In Fig. 6, a degenerate Hopf bifurcation curve (Hopf curve plus first Lyapunov index equal to zero, i.e. codimension 1 bifurcation) is depicted. Notice that the degenerate Hopf bifurcation of codimension 2 (regular Hopf plus the first and second Lyapunov indexes set equal to zero) is very close to the parameter setting depicted here. Also, it is very interesting to note that the Hopf degeneracy curve of codimension 1 has a turning point close to the limiting point in which $c_3 \to 0$. This type of bending of this Hopf degeneracy has been observed before in other systems (see [Planeaux, 1993; Moiola & Chen, 1996] for more details) in connection with the appearance of degenerate Hopf bifurcations of codimension 2 regarding multiple cycles.

In Figs. 7 and 8 similar structures of stable and unstable limit cycles are depicted. After

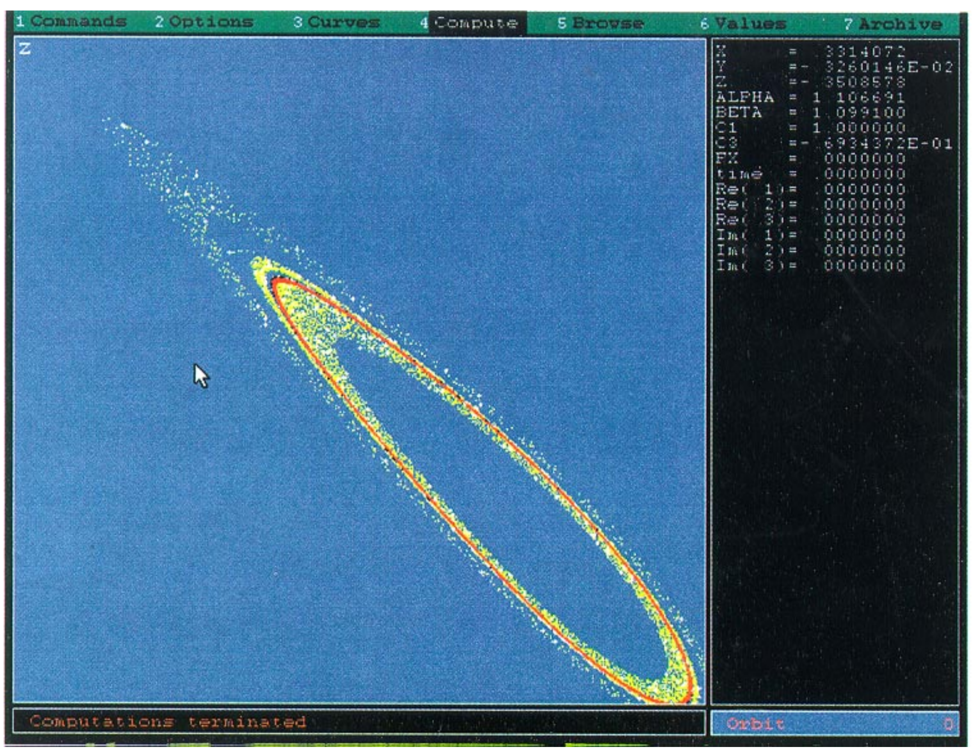


Fig. 1. Stable limit cycle (in red for color picture) surrounding the equilibrium point $P^-$ for $\alpha = 1.106691$, $\beta = 1.0991$, and $c_3 = -0.06934372$. (The limits of the axes are: $x_{\min} = 0$, $x_{\max} = 0.35$; $z_{\min} = -0.35$, $z_{\max} = 0$.)

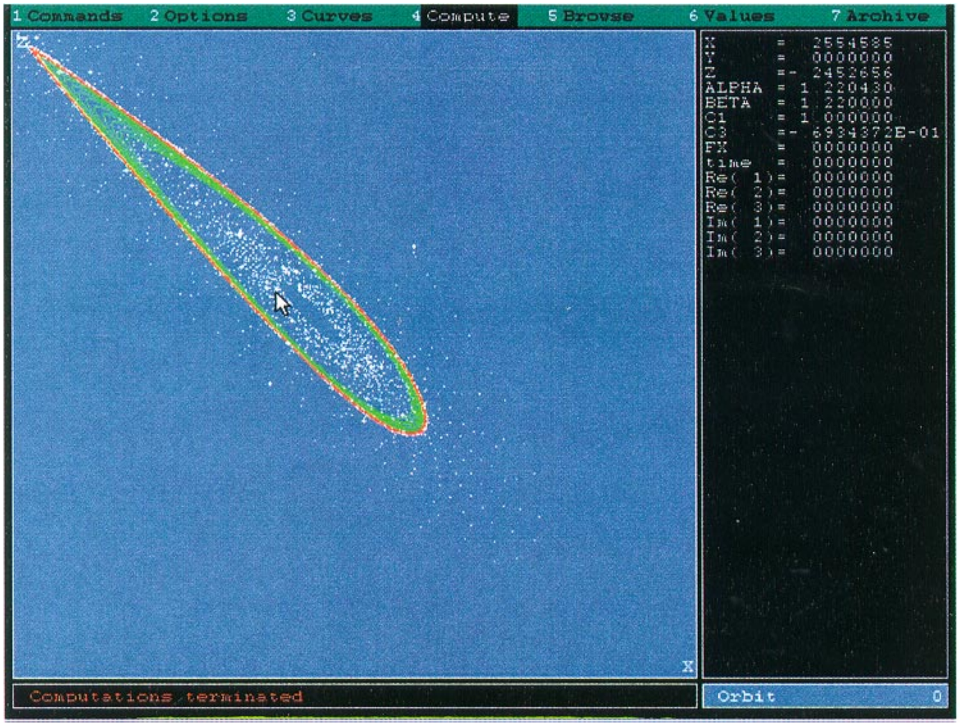


Fig. 2. Stable limit cycle (in red for color picture) surrounding the equilibrium point $P^{-}$ for $\alpha = 1.22043$, $\beta = 1.22$, and $c_3 = -0.06934372$. (The limits of the axes are: $x_{\min} = 0$, $x_{\max} = 0.60$; $z_{\min} = -0.60$, $z_{\max} = 0$.)


Fig. 3. Stable limit cycle (one trajectory in red for color picture) surrounding the three equilibria for $\alpha = 0.9893846$, $\beta = 0.96927$, and $c_3 = -0.06934372$. The unstable limit cycles — denoted by white continuous circles — surrounds the equilibria $P^{\pm}$. (The limits of the axes are: $x_{\min} = -0.5$, $x_{\max} = 0.5$; $z_{\min} = -0.5$, $z_{\max} = 0.5$.)

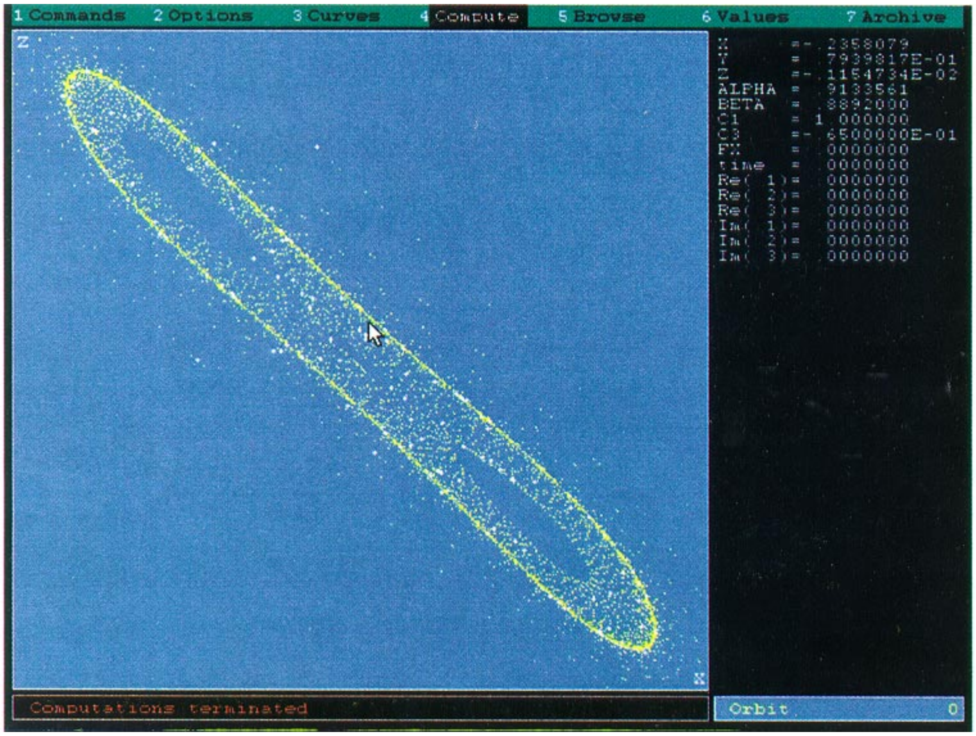


Fig. 4. Stable limit cycle (one trajectory in yellow for color picture) surrounding the three equilibria for $\alpha = 0.9133561$, $\beta = 0.88920$, and $c_3 = -0.065$. The unstable limit cycles are not shown in the figure. (The limits of the axes are: $x_{\min} = -0.5$, $x_{\max} = 0.5$; $z_{\min} = -0.5$, $z_{\max} = 0.5$.)

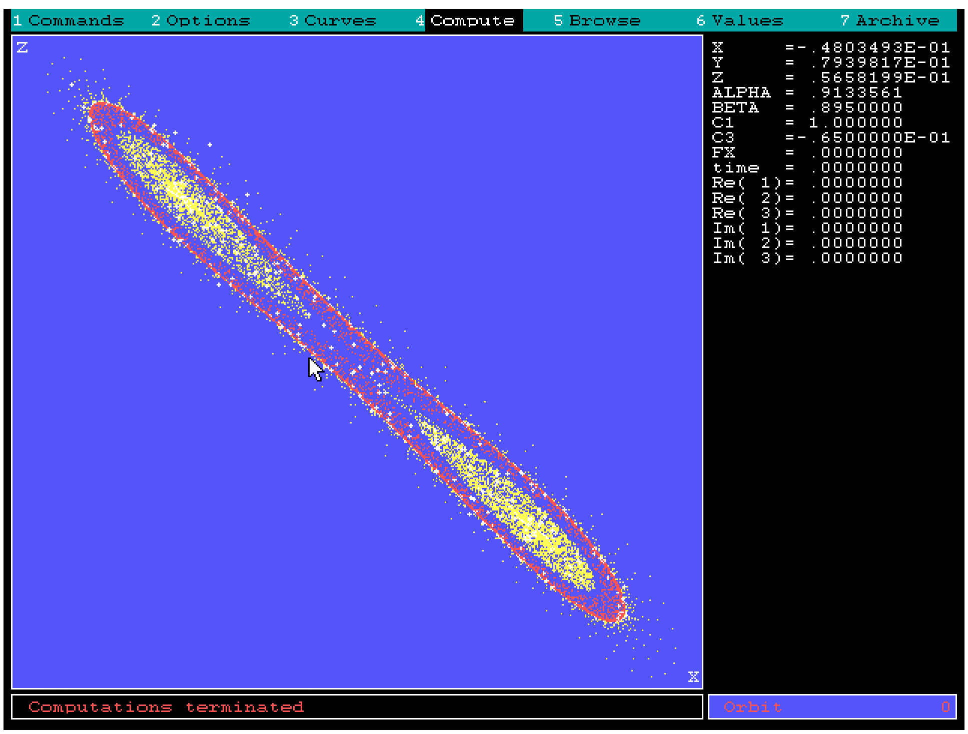


Fig. 5. Stable limit cycle surrounding the three equilibria for $\alpha = 0.9133561$, $\beta = 0.8950$, and $c_3 = -0.065$. The unstable limit cycles are not shown in the figure but they are closer to the stable limit cycle compared to the situation shown in Fig. 4. (The limits of the axes are: $x_{\min} = -0.5$, $x_{\max} = 0.5$; $z_{\min} = -0.5$, $z_{\max} = 0.5$.) (For color picture: yellow and white dots indicate contracting trajectories; red dots indicate expanding trajectories.)

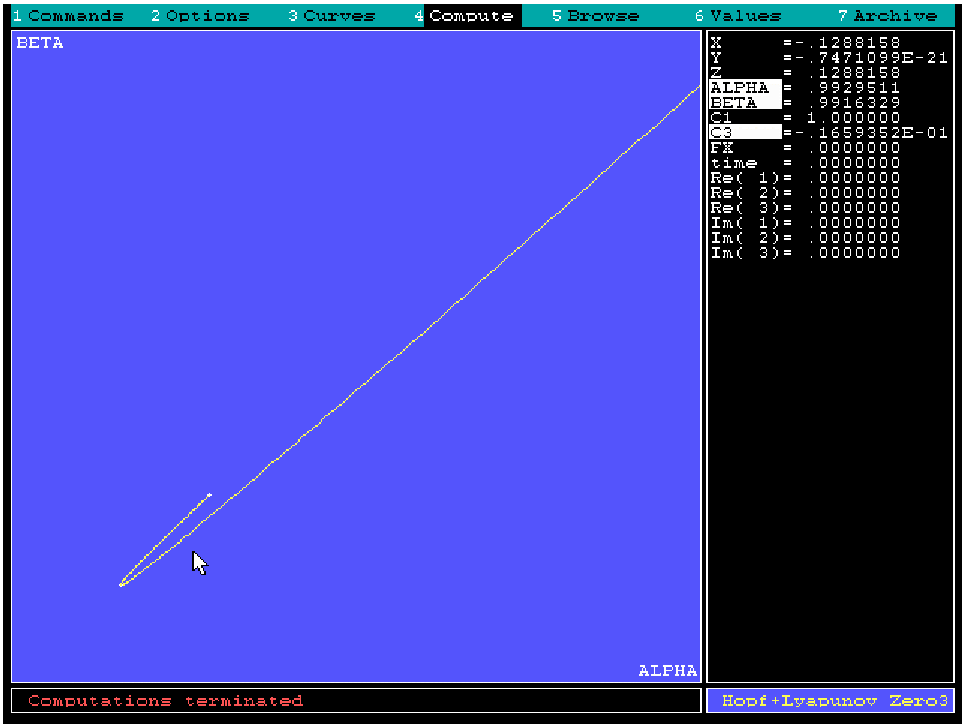


Fig. 6. Degenerate Hopf bifurcation curve (Hopf plus first Lyapunov index equal to zero). On this curve, on its top right there is a point having an extra condition: The second Lyapunov index is also equal to zero (not shown in the figure). (The limits of the axes are: $\alpha_{\min} = 0.98$, $\alpha_{\max} = 1.05$; $\beta_{\min} = 0.98$, $\beta_{\max} = 1.05$.)

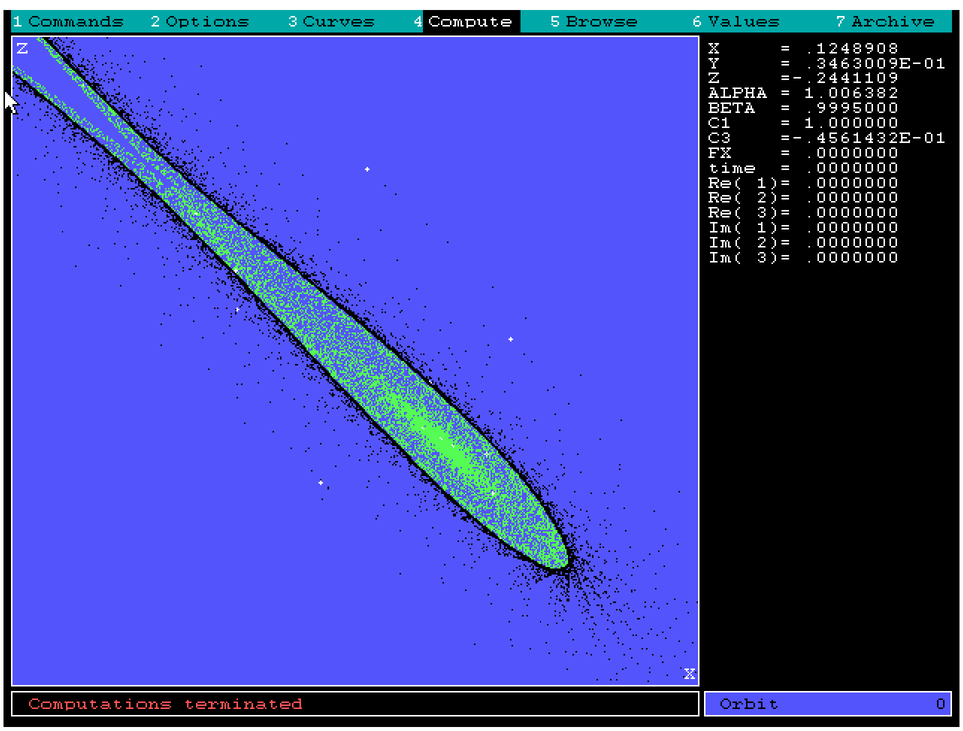


Fig. 7. Stable limit cycle surrounding the three equilibria for $\alpha = 1.006382$, $\beta = 0.9995$, and $c_3 = -0.04561432$ (in black for color picture). The unstable limit cycles surround the symmetrical equilibria. (The limits of the axes are: $x_{\min} = -0.1$, $x_{\max} = 0.4$; $z_{\min} = -0.4$, $z_{\max} = 0.1$.)

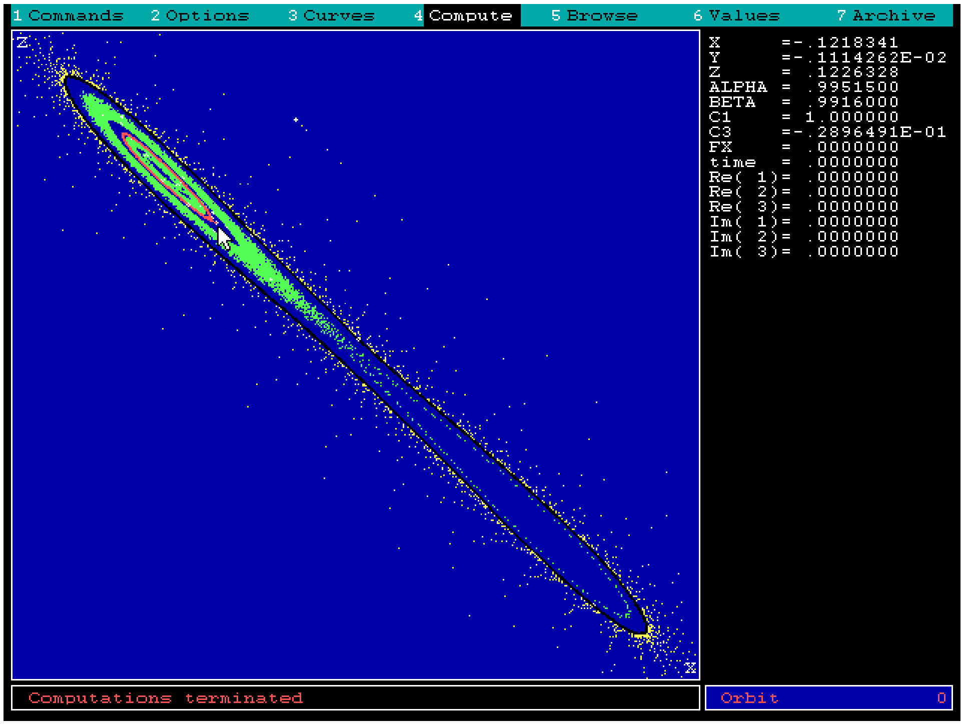


Fig. 8. Stable limit cycle surrounding the three equilibria for $\alpha = 0.99515$, $\beta = 0.9916$, and $c_3 = -0.02896491$ (in black for color picture). The unstable limit cycles surround the symmetrical equilibria (in red for color picture). (The limits of the axes are: $x_{\min} = -0.3$, $x_{\max} = 0.3$; $z_{\min} = -0.3$, $z_{\max} = 0.3$.)

looking for more complex structures of multiple limit cycles (excluding period-doubling bifurcations and chaotic attractors), we observed that the multiple cycles (more than two nested cycles) belong to a narrow band in the parameter setting, as it was observed before in other systems [Planeaux, 1993; Moiola & Chen, 1996], using AUTO [Doedel, 1986]. Since the accuracy of computer simulations using LOCBIF is limited, AUTO software should be used for this task.

## 3. Discussions and Concluding Remarks

The formulas for the amplitude equations of the limit cycles arising in Chua's circuit are obtained using a frequency domain approach. Moreover, an expression of the first Lyapunov index, which determines the stability of the emerging periodic solutions, is computed in terms of the system parameters. This preliminary step is important in order to study multiple limit cycles arising from degenerate Hopf bifurcations in this circuit. This frequency domain approach (also called Graphical Hopf Theorem, GHT for short) for analyzing periodic solutions has provided very useful results in dealing with degenerate Hopf bifurcations [Moiola & Chen, 1996]. Very recently, this approach has been adapted to handle approximate detection of the first period-doubling bifurcation [Berns *et al.*, 1998] in the time-delayed version of Chua's circuit, by considering a higher-order expansion of the periodic solutions. Other related research using a unified formulation for both Hopf and period-doubling bifurcations were given by Rand [1989] and Belhaq and Houssni [1995] for a specific system, and Tesi *et al.* [1996] and Basso *et al.* [1997] for a broader class of nonlinear systems. Continuous efforts and progress made by some researchers from mechanical engineering [Szemplinksa-Stupnicka & Rudowski, 1993; Donescu & Virgin, 1996; Janicki & Szemplinksa-Stupnicka, 1997] in characterizing accurately the birth of period-doubling bifurcations (subharmonic resonances in forced systems) using similar methods, had encouraged us to pursue a simple methodology to handle several types of periodic solutions and their bifurcations. Since the GHT provides also a useful graphical interpretation, it seems natural to continue this effort in order to give better accurate results using higher-order harmonic balance approximations and a type of convergence test for the accuracy of the solutions. A unified approach to treat both Hopf bifurcations and period-doubling bifurcations as well as indication of symmetry-breaking would be very useful. In such studies, Chua's circuit dynamics offers an excellent vehicle and paradigm for testing any future development in this area.

## Acknowledgments

J. L. Moiola acknowledges the partial support of a Fulbright-Antorchas grant as well as CONICET and UNS. He also acknowledges the hospitalitiy of the Department of Electrical Engineering and Computer Sciences at UCB. The authors would like to thank D. W. Berns for the computation of the critical point having the vanishing of the first and second Lyapunov indexes. Additional support is provided by ONR grant number N00014-97-1-0463.

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