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*Chua's Circuit and the Qualitative Theory of Dynamical Systems**

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ABSTRACT: *Simple electronic oscillators were at the origin of many studies related to the qualitative theory of dynamical systems. Chua's circuit is now playing an equivalent role for the generation and understanding of complex dynamics.* © 1997 The Franklin Institute. Published by Elsevier Science Ltd

1. Oscillating Circuits and the Origin of the Qualitative Theory

In the nineteenth century, Joseph Fourier wrote: "The study of Nature is the most productive source of mathematical discoveries. By offering a specific objective, it provides the advantage of excluding vague problems and unwieldy calculations. It is also a means to form the Mathematical Analysis, and isolate the most important aspects to know and to conserve. These fundamental elements are those which appear in all natural effects."

The important development of the theory of dynamic systems, during this century, has essentially its origins in the study of the 'natural effects' encountered in systems of mechanical, electrical, or electronic engineering, and the rejection of non-essential generalizations. Most of the results obtained in the abstract dynamic systems field have been possible on the foundations of results from the concrete dynamic systems field. It is also worth noting that the majority of scientists (including mathematicians) were not led to their discoveries by a process of deduction from general postulates or general principles, but rather by a thorough examination of properly chosen particular cases, and observation of concrete processes. The generalizations have come later, because it is far easier to generalize an established result than to discover a new line of argument.

Since Andronov (1932), traditionally three different approaches have been used for the study of dynamical systems (**26**): qualitative methods, analytical methods and numerical methods. To define the 'strategy' of qualitative methods, one has to note

* In honour of my friend Leon Chua on his sixtieth birthday.

that the solutions of equations of non-linear dynamic systems are in general non-classical transcendental functions of the mathematical analysis, which are very complex. This 'strategy' is of the same type as that used for the characterization of a function of a complex variable by its singularities: zeros, poles, essential singularities. Here, the complex transcendental functions are defined by the singularities of continuous (or discrete) dynamic systems such as: stationary states which are equilibrium points (fixed points) or periodical solutions (cycles), which can be stable or unstable; trajectories (invariant curves) passing through saddle singularities of two-dimensional systems; the stable and unstable manifold for a dimension greater than two; the boundary, or separatrix, of the influence domain (domain of attraction, or basin) of a stable (attractive) stationary state; homoclinic and heteroclinic singularities; more complex singularities of fractal or nonfractal type.

The qualitative methods consider the nature of these singularities in the phase (or state) space, and their evolutions when parameters of the system vary, or in the presence of a continuous structure modification of the system (study of the bifurcation sets in the parameter space, or in a function space) (**4–6**).

In fact, initially, qualitative methods developed from the fundamental studies of circuits in radio engineering. Indeed, in 1927, Andronov, the most famous of Mandelstham's students, defended his thesis with the topic formulated by Mandelstham, *The Poincaré's limit cycles and the theory of oscillations*. This thesis is a first-rank contribution to the evolution of the theory of non-linear oscillations, because it opens a new method of applications for Poincaré's qualitative theory of differential equations, with many practical consequences. With this work, Andronov was the first to see that the phenomena of free (or self) oscillations, for example, that generated by the Van der Pol oscillator, correspond to limit cycles. It is from the study of oscillators that Andronov (**4**) later amplified his activity with a precise purpose: the development of a theory of non-linear oscillations, to make use of mathematical tools common to different scientific disciplines.

Andronov and Pontrjagin formulated in 1937 the necessary and sufficient conditions of structural stability for autonomous two-dimensional systems. These conditions are: the system has only a finite number of equilibrium points and limit cycles, which are not in a critical case in the Lyapunov sense; no separatrix joins the same or two distinct saddle points. In this case, it is possible to define, in the parameter space of the system, a set of cells inside which the same qualitative behaviour is preserved (**4**).

The knowledge of such cells is of major importance for the analysis and synthesis of dynamic systems in physics or engineering. On the boundary of a cell, the dynamic system is structurally unstable, and for autonomous two-dimensional systems (two-dimensional vector fields), structurally stable systems are dense in the function space. Until 1966, the conjecture of the extension of this result to higher-dimensional systems was generally supposed to be true.

Andronov also extended the notion of structural stability for dynamic systems described by

$$dx/dt = f(x,y), \quad \mu dy/dt = g(x,y), \quad \mu > 0 \tag{1}$$

where x and y are vectors, μ is a 'small' parameter vector representing the parasitic elements of the system, and $f(x,y)$, and $g(x,y)$ are bounded and continuous in the domain of interest of the phase space (**4**).

If $\mu = 0$, Eq (1) reduces to a system of lower dimension:

$$\mathrm{d}x/\mathrm{d}t = f(x,y), \quad g(x,y) = 0 \tag{2}$$

For theoretical, as well as practical purposes, a fundamental problem consists in determining when the 'small' terms $\mu \mathrm{d}y/\mathrm{d}t$, representing the effects of the parasitic elements (small capacitances and inductances in an electrical system, small damping and inertia in a mechanical one) are negligible. In other words, when is the motion described by Eq (1) sufficiently close to that described by Eq (2) so that it can be represented by the solution of Eq (2) defined for a lower dimension?

It is interesting to note that the formulation of this important problem has its origin in a discussion (1929) between Andronov and Mandelstham, related to the one time-constant electronic multivibrator. Without considering the parasitic elements, such as parasitic capacitances and inductances, the multivibrator is nominally described by a first-order (one-dimensional) autonomous differential equation, such as Eq (2), where x is now a scalar (voltage). If it is required that $y(t)$ be a continuous function of time, then it was shown by Andronov that Eq (2) does not admit any non-constant periodic solution. Such a mathematical result is contrary to physical evidence, because the one time-constant multivibrator is known to oscillate with a periodic waveform. In the Mandelstham–Andronov discussion of this paradox, the following alternative was formulated: (a) either the nominal model Eq (2) is not appropriate to describe the practical multivibrator, or (b) it is not being interpreted in a physically significant way.

Andronov has shown that either term of the alternative may be used to resolve the paradox, provided the space of the admissible solutions is properly defined. In fact, specifying that the solutions must be continuous and continuously differentiable leads to the conclusion that Eq (2) is inappropriate on physical grounds, because the real multivibrator possesses several small parasitic elements. This then leads to a model in the form of Eq (1), the vector μ being related to the parasitic elements. However, Eq (1) appears unsatisfactory from a practical point of view. Indeed, the existence and the stability of the required periodic solution depends not only on the presence of parasitic parameters, which are difficult to measure in practice, but also on their relative magnitudes. Andronov has shown that the strong dependence on parasitic elements can be alleviated by means of the second term of the alternative. This is done by generalizing the set of admissible solutions, defined now as consisting of piecewise continuous and piecewise differentiable functions. Then the first-order differential Eq (2) is supplemented by some 'jump' conditions (called Mandelshtam conditions) permitting the joining of the various pieces of the solution, which now can be periodic. The theory of models having the form of Eq (1) associated with the problem of dimension reduction, and that of relaxation oscillators (4), began with this study.

2. *Chua's Circuit and the Contemporary Qualitative Theory*

One of the reasons for the popularity of Chua's circuit is that it can generate a large variety of complex dynamics, and convoluted bifurcations, from a simple model in the form of a three-dimensional autonomous piecewise linear ordinary differential equation (flow). It concerns a concrete realization (with discrete electronic components, or implemented in a single monolithic chip), whereas the very well-known Lorenz equation, which is also a three-dimensional flow, is related to a very rough low-dimensional model of atmospheric phenomena, far from the real complexity of 'nature'.

As mentioned above, until 1966, an extension of two-dimensional structural stability conditions, for dimensions higher than two, was conjectured. However, Smale (**24, 25**) showed that this conjecture is false in general. So, it appears that, with an increase of the system dimension, one has an increase of complexity of the parameter (or function) space. The boundaries of the cells defined in the phase space, as well as in the parameter space, have in general a complex structure, which may be a fractal (self-similarity properties) for n-dimensional vector fields, $n>2$.

Sufficient conditions for structural stability were formulated by Smale (**23**). A system is structurally stable when the fixed (equilibrium) points and periodic solutions (orbits) are structurally stable and of finite number, when the set of non-wandering points consists of these stationary states only, and when all the stable and unstable manifolds intersect transversally. Such systems are now known as Morse–Smale systems.

The analysis of bifurcations, which transform a Morse–Smale system into a system having an enumerable set of periodic orbits, has been a favourite topic for research since 1965. There certainly exist many such bifurcations of different types. Gavrilov, Afraimovitch and Shilnikov (**1–3, 10–14, 20–22**) have studied some of them, related to the presence of structurally unstable homoclinic or heteroclinic curves associated with an equilibrium point, or a periodic orbit for a dimension $m>3$. Their results have contributed to the study of the popular Lorenz differential equation ($m = 3$) by Afraimovitch *et al.* (**1**). Chua's circuit belongs to the class of three-dimensional 'continuous' dynamical systems (flows with $m = 3$). With respect to other studies, it (**1**) has the advantage of exhibiting 'physical' bifurcations which transform a Morse–Smale system into a system having an enumerable set of periodic orbits.

Let us limit our discussion to this class of three-dimensional 'continuous' dynamical systems (flows), and two-dimensional diffeomorphisms associated with them from a Poincaré section. Newhouse (**19**) formulated a very important theorem stating that in any neighbourhood of a C^r-smooth ($r\geq 2$) dynamical system, in the space of discrete dynamical systems (diffeomorphisms), there exist regions for which systems with homoclinic tangencies (then with structurally unstable or nonrough homoclinic orbits) are dense. Domains having this property are called Newhouse regions. This result is completed in (**12**), which asserts that systems with infinitely many homoclinic orbits of any order of tangency, and with infinitely many arbitrarily degenerate periodic orbits, are dense in the Newhouse regions of the space of dynamical systems. This has an important consequence: systems belonging to a Newhouse region are such that a complete study of their dynamics and bifurcations is impossible. In this case, only particular characteristics of such systems can be studied, such as the presence of nontrivial hyperbolic subsets (infinite number of saddle cycles). Let us restrict our discussion to a one-parameter family of three-dimensional dynamical systems leading to Newhouse intervals, and the associated family of two-dimensional diffeomorphisms (differentiable invertible maps). In such intervals there are dense systems with an infinite number of stable cycles (periodic orbits) if the modulus of the product of their multipliers (eigenvalues) is less than one, and with infinitely many totally unstable cycles if this modulus is higher than one (**22**). This last result furnishes a theoretical foundation for the fact that many of the attractors studied contain a 'large' hyperbolic subset in the presence of a finite or infinite number of stable cycles (**18**). Generally, such stable cycles have large periods, and narrow 'oscillating' tangled basins, which are difficult to determine numerically.

Systems having infinitely many unstable periodic orbits (they are not of Morse–Smale type) give rise either to strange attractors or to strange repellors. Strange repellors are at the origin of two phenomena: either that of a chaotic transient toward only one attractor for small changes of initial conditions, or that of fuzzy (or fractal) boundaries (**15**) separating the basins of several attractors. In fact, a fractal basin boundary also gives rise to chaotic transients, but toward at least two attractors in the presence of very small variations of initial conditions. The structure identification of strange attractors and repellors, and the bifurcations giving rise to such a complex dynamics, constitute one of the most important current problems.

Strange attractors are at present divided into three principal classes: hyperbolic, Lorenz-type and quasi-attractors (**22**).

Hyperbolic attractors are the limit sets for which Smale's Axiom A is satisfied, and are structurally stable. Periodic orbits and homoclinic orbits are dense and are of the same saddle type, that is, the stable (or unstable) manifolds of all the trajectories have the same dimension. In particular, this is the case for Anosov systems and the Smale–Williams solenoid. Until now, it seems that such attractors have not been found in concrete applications.

Lorenz attractors are not structurally stable, though their homoclinic and heteroclinic orbits are structurally stable (hyperbolic). They are everywhere dense, and no stable orbit appears under small parameter variations (**1**) (for more references, see also Shilnikov (**22**)). Both hyperbolic and Lorenz attractors are stochastic, and thus can be characterized from the ergodic theory.

Quasi-attractors (an abbreviation of 'quasistochastic attractors'(**2**); for more references, see also (**22**)) are not stochastic, and are more complex than the two above attractors. A quasi-attractor is a limit set enclosing periodic orbits of different topological types (e.g. stable and saddle periodic orbits), which are structurally unstable orbits. Such a limit set may not be transitive. Attractors generated by Chua's circuits (**7**, **8**) associated with saddle-focus homoclinic loops are quasi-attractors. For three-dimensional systems, mathematically, such attractors should contain infinitely stable periodic orbits (**18**), a finite number of which can only appear numerically owing to the finite precision of computer experiments. They coexist with nontrivial hyperbolic sets. Such attractors are encountered in many models, such as the Lorenz attractor, the spiral-type and the double-scroll attractor generated by Chua's circuit, or the Hénon map, for certain domains of the parameter space.

The complexity of a quasi-attractor is essentially due to the existence of structurally unstable homoclinic orbits (homoclinic tangencies) not only in the system itself, but also in any system close to it. This results in a sensitivity of the attractor structure to small variations of the parameters of the generating dynamical equation, i.e. quasi-attractors are structurally unstable. Such systems belong to Newhouse regions, with the consequences given above.

In the n-dimensional case, $n > 3$, the situation becomes more complex, and first results (in particular, a theorem showing that a system can be studied in a manifold of lower dimension) have been given by Gonchenko *et al.* (**12**, **14**).

In addition to its interest in engineering applications, Chua's circuit generates a large number of complex fundamental dynamical phenomena. Indeed, it is the source of different bifurcations giving rise to chaotic behaviours (period doubling cascade, breakdown of an invariant torus, etc.). The corresponding attractors are related to complex homoclinic or heteroclinic structures. One of these attractors, the double scroll, characterized by the presence of three equilibrium points of saddle-focus type, arises from two nonsymmetric spiral attractors. It is different from other known attractors of autonomous three-dimensional systems in the sense that it is multistructural.

3. *Conclusion*

An important book (**17**) collects many contributions devoted to applied and theoretical questions related to this circuit, which since that publication has given rise to many new developments. The synchronization of chaotic signals generated by Chua's circuit has led to an increasing number of publications, with applications for secure communications (**16**). Moreover, a wide field of research has been initiated by using a two- and three-dimensional grid of resistively coupled Chua's circuits. From such networks, waves and spatio-temporal chaos can be considered with travelling, spiral, target or scroll waves (**9**). Here, Chua's circuit is used as the basic cell in a discrete cellular neural network (CNN).

The study of quasi-attractors (which are generated in particular by Chua's circuit) is only beginning, and so there is a wide field for research. Such attractors cannot be made structurally stable via any finite parameter unfolding of the corresponding system. Arbitrarily small variation of parameters can lead to significant change of the attractor structure. This results in the impossibility of attaining a complete description of their dynamics and their bifurcation space. Even for three-dimensional flows the results are not complete. A fortiori the extension to higher-dimensional cases is a source of problems that remain for the future, because it is not trivial and provides the occasion to consider new dynamical phenomena (**22**). Chains of Chua's circuits may be of value in such research. Nevertheless, a complete study of such processes being impossible, future research will only be concerned with some specific and typical properties of systems generating quasi-attractors. Related to the above question is the problem of formulation of a good model (**10**), which has a sufficient number of parameters to analyse all possible bifurcations of the steady states, homoclinic and heteroclinic structures, etc. Applied aspects of quasi-attractors have been mentioned by Shilnikov (**22**). These relate to the development of associative memories, and an approach for understanding the memory mechanisms. As indicated in Section 1, simple electronic oscillators were at the origin of many studies related to the qualitative theory of dynamical systems. It appears that Chua's circuit is now playing an equivalent role for the generation and understanding of complex dynamics, in relation to many applications.

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