Then

$$K' = \sum_{n=0}^{\infty} x'(n) x'^T(n) \quad (21)$$

$$x'(0) = 0 \quad (22)$$

$$x'(n) = (I - C)^{-1} E A^{n-1} b, \qquad \text{for } n > 0 \quad (23)$$

and

$$K' = \left[(I-C)^{-1}b\right]\left[(I-C)^{-1}b\right]^T + (I-C)^{-1}EKE^T(I-C)^{-T} \quad (24)$$

where the superscript $-T$ indicates inverse transpose. The matrix K' can thus be calculated from K.

The calculation of W' for the extended filter is not as straightforward. To begin, define

$$\delta(i) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \{1\} \\ \{2\} \\ \\ \{i\} \\ \\ \\ \{m\} \end{matrix} . \quad (25)$$

The (i, j)th element of the matrix W' is defined by

$$\{W'\}_{i,j} = \sum_{n=0}^{\infty} y^{(i)}(n) y^{(j)}(n) \quad (26)$$

where $y^{(i)}(n)$ is the output at time n from an initial condition

$$x'(0) = \delta(i). \quad (27)$$

It follows that

$$x'(n) = \left[(I-C)^{-1}ED\right]^n \delta(i) \quad (28)$$

$$y^{(i)}(n) = g\left[(I-C)^{-1}ED\right]^n \delta(i) \quad (29)$$

and

$$W' = g^T g + D^T W D \quad (30)$$

where

$$W = \sum_{n=0}^{\infty} \left[g(I-C)^{-1}E\left(D(I-C)^{-1}E\right)^n\right]^T \cdot g(I-C)^{-1}E\left(D(I-C)^{-1}E\right)^n. \quad (31)$$

The matrix W can be calculated from the same algorithm as K, but starting with the matrices $[D(I-C)^{-1}E]^T$ and $[g(I-C)^{-1}E]^T$ instead of A and b, respectively.

Summary

A method for calculating the matrices K' and W' corresponding to an extended digital filter has been demonstrated. The algorithm can be used to calculate the quantization noise generated at nodes other than the storage nodes of the filter.

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A Chaotic Attractor from Chua's Circuit

T. MATSUMOTO

Abstract **—A chaotic attractor has been observed with an extremely simple autonomous circuit. It is third order, reciprocal and has *only one nonlinear element; a 3-segment piecewise-linear resistor.* The attractor appears to have interesting structures that are different from Lorenz's and Rössler's.**

Our purpose here is to report that a chaotic attractor has been observed with an extremely simple autonomous circuit. It is third-order, reciprocal, and has only one nonlinear element; *a 3-segment piecewise-linear resistor.* It is a simplified version of a circuit suggested by Leon Chua of the University of California, Berkeley, who was visiting Waseda University, Japan, during October 1983–January 1984.

Consider the circuit of Fig. 1(a) where the constitutive relation of the nonlinear resistor is given by Fig. 1(b). The dynamics is described by

$$C_1 \frac{dv_{C_1}}{dt} = G(v_{C_2} - v_{C_1}) - g(v_{C_1})$$

$$C_2 \frac{dv_{C_2}}{dt} = G(v_{C_1} - v_{C_2}) + i_L$$

$$L\frac{di_L}{dt} = -V_{C_2} \quad (1)$$

where v_{C_1}, v_{C_2}, and i_L denote voltage across C_1, voltage across C_2, and current through L, respectively. Fig. 2 shows the chaotic

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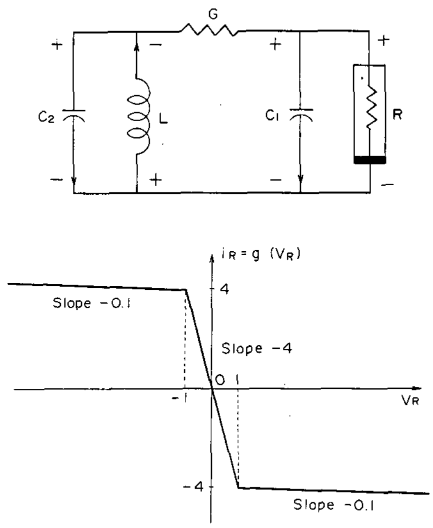

Fig. 1.

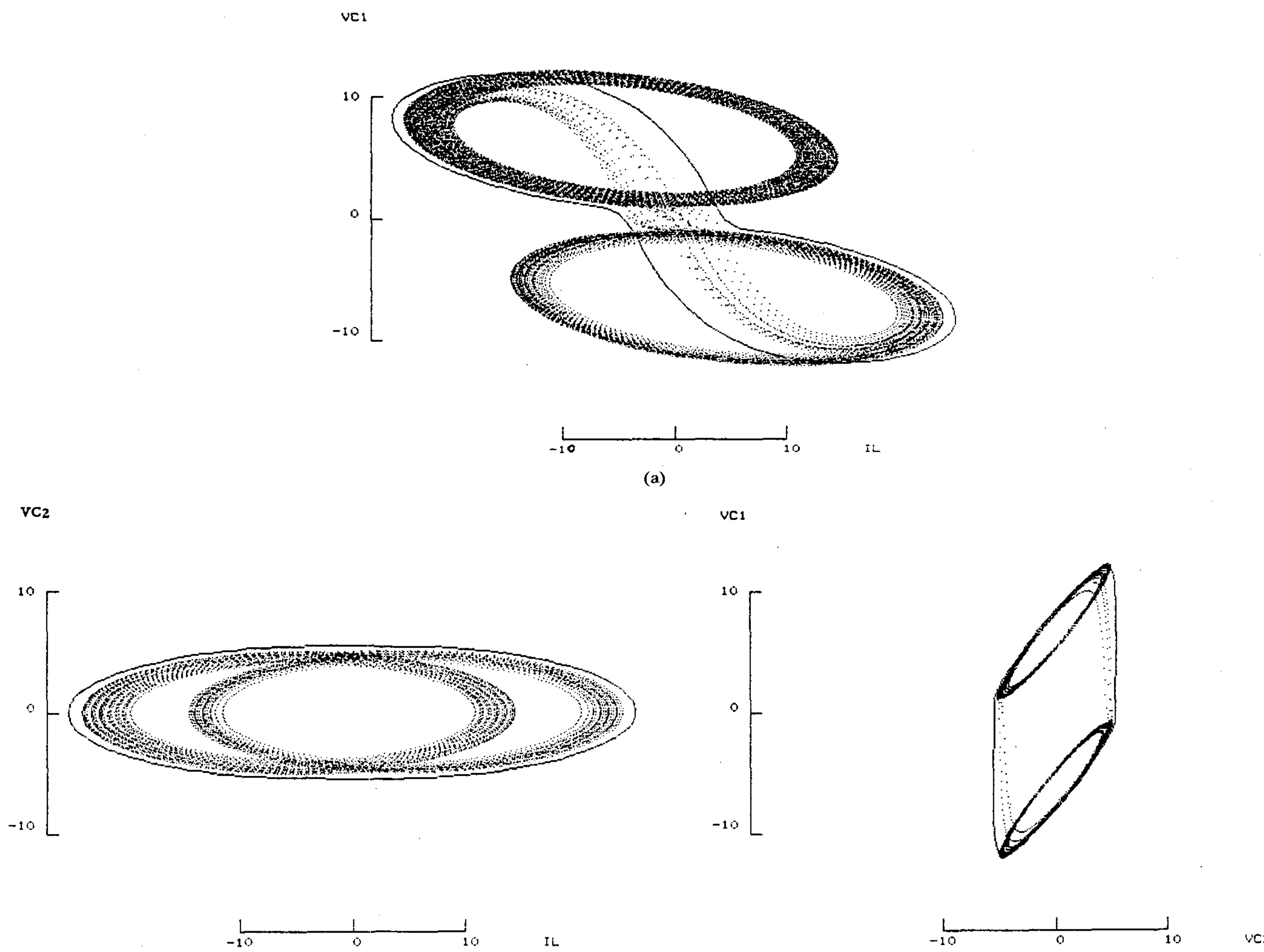

Fig. 2.

attractor observed by solving (1) with

$$1/C_1 = 10, \quad 1/C_2 = 0.5, \quad 1/L = 7, \quad G = 0.7. \tag{2}$$

Fig. 2(a)–(c) are the projections of the attractor onto the (i_L, v_{C_1})-plane, (i_L, v_{C_2})-plane and (v_{C_2}, v_{C_1})-plane, respectively. (The fourth-order Runge–Kutta was used with step size 0.02). It is interesting to observe that a saddle-type *hyperbolic* periodic orbit (*not* a stable limit cycle) is present outside the attractor. (Newton iteration was used).

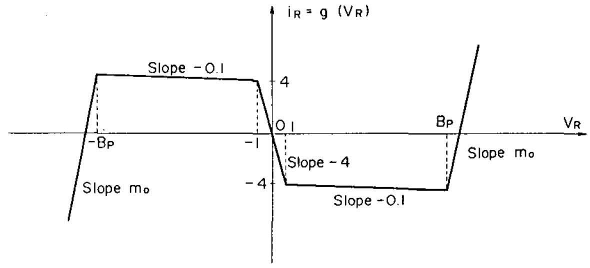

Fig. 3.

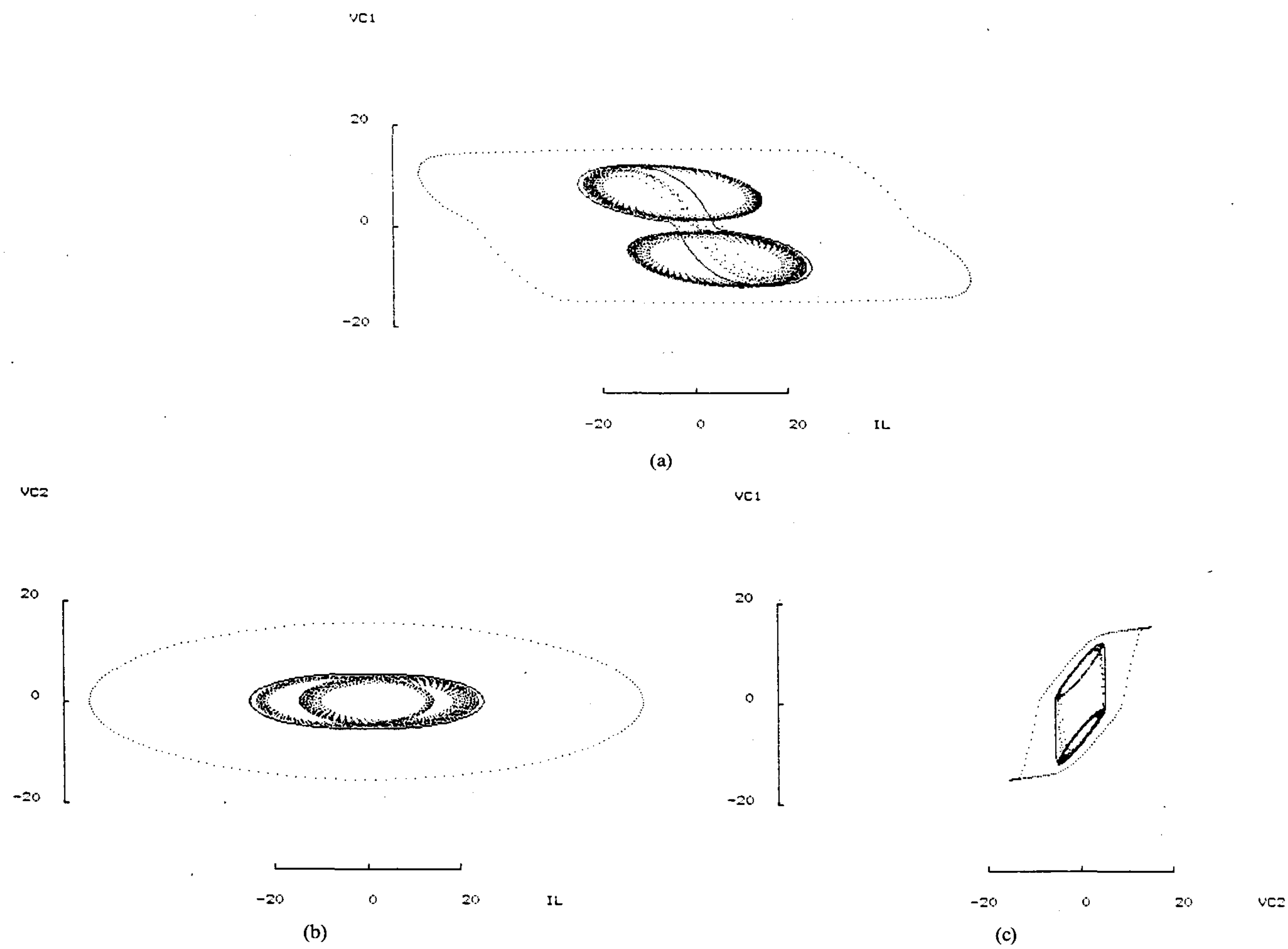

Fig. 4.

If the reader feels uncomfortable with the function g of Fig. 1(b) in that it is not eventually passive and there are initial conditions with which (1) diverges, he can simply replace Fig. 1(b) with Fig. 3. If $B_p = 14$, it has *no effect* on the attractor *and* on the hyperbolic periodic orbit, because $|v_{C_1}(t)| < 14$ for all $t \geq 0$ on the attractor and on the hyperbolic periodic orbit. The only difference is the appearance of a large *stable* limit cycle, as shown in Fig. 4, where (1) does not diverge with any initial condition ($B_p = 14$, $m_0 = 5$). There are three initial conditions in Fig. 4;

(i)

$$V_{C_1}(0) = 1.45305$$
$$V_{C_2}(0) = -4.36956$$
$$i_L(0) = 0.15034$$

for the chaotic attractor,

(ii)

$$V_{C_1}(0) = 9.13959$$
$$V_{C_2}(0) = -1.35164$$
$$i_L(0) = -59.2869$$

for the large stable limit cycle, and

(iii)

$$V_{C_1}(0) = 10.00717$$
$$V_{C_2}(0) = 1.80100$$
$$i_L(0) = -23.90375$$

for the hyperbolic periodic orbit with period $T = 3.93165$.

It has been observed that the attractor *persists* for at least the following parameter ranges:

(i) $7.2 \leqslant 1/C_1 \leqslant 11.5$, when $1/C_2 = 0.5$, $1/L = 7$ and $G = 0.7$ are fixed,

(ii) $0.3 \leqslant 1/C_2 \leqslant 0.8$, when $1/C_1 = 10$, $1/L = 7$ and $G = 0.7$ are fixed,

(iii) $5.8 \leqslant 1/L \leqslant 11$, when $1/C_1 = 10$, $1/C_2 = 0.5$ and $G = 0.7$ are fixed, and

(iv) $0.52 \leqslant G \leqslant 0.8$, when $1/C_1 = 10$, $1/C_2 = 0.5$ and $1/L = 7$ are fixed.

The attractor appears to have interesting structures different from Lorenz's [1] and Rössler's [2]. Let us briefly describe some of the differences. Equation (1) has three equilibria; one at the origin, one in the half space $V_{C_1} > 0$ and another in the half space $V_{C_1} < 0$. The equilibrium at the origin has one *positive* real eigenvalue and a pair of *complex*-conjugate eigenvalues with *negative* real part. Other equilibria have one *negative* real eigenvalue and a pair of complex-conjugate eigenvalues with positive real part. Now a detailed analysis which will be reported shortly shows that the origin belongs to the attractor as an invariant set. Recall that for the Lorenz equation, this is also the case. The origin for the Lorenz equation, however, is a saddle, i.e., all eigenvalues are real. Another difference is that (1) is symmetric with respect to the origin while the Lorenz equation is symmetric with respect to the z-axis. It is known that the Rössler attractor [2] does not contain any equilibrium. Hence, it is different from ours.

One of the reviewers pointed out that chaotic attractors have been observed in feedback systems with piecewise-linear feedback characteristic [3], [4]. The attractor in [3] does not appear to contain any equilibrium as long as the pictures show. Note also that the equilibrium in the middle has one *negative* real eigenvalue and a pair of complex conjugate eigenvalues with *positive* real part. Hence it is different from ours. The dynamics considered in [4] appears to have the same singularity types as that of (1). It is not clear yet, however, how the attractor in [4] is related to ours since detailed analyses are not reported in [4]. Circuit theoretically, however, there is one definite distinction between (1) and equations in [1]–[4]; the circuit of Fig. 1 has no coupling elements and hence *reciprocal* [5], while the systems in [1]–[4] are *nonreciprocal* and, therefore, they cannot be realized by reciprocal circuits.

Many more interesting structures have been observed with parameter values different from (2). Details including geometric structure, Lyapunov exponents, bifurcations and circuit realizations will be reported in later papers.

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A Stability Inequality for Nonlinear Discrete-Time Systems with Slope-Restricted Nonlinearity

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Abstract—A novel frequency domain criterion is presented for the absolute stability of Lure type single-input single-output discrete-time systems with slope-restricted nonlinearity.

I. Introduction

The purpose of this paper is to present a frequency-domain criterion for the asymptotic stability in the large (ASIL) of a discrete-time system characterized by a stable linear part $G(z)$:

$$G(z) = \frac{h_n z^{n-1} + h_{n-1} z^{n-2} + \cdots + h_1}{z^n + a_n z^{n-1} + \cdots + a_1} \tag{1a}$$

having output $y(r)$ and input $-f(y(r))$, with the restrictions

$$f(0) = 0, \quad 0 \leqslant \frac{f(y(r))}{y(r)} \leqslant k, \quad -k_1 \leqslant \frac{\Delta f(y(r))}{\Delta y(r)} \leqslant k_2,$$
$$k_2 \geqslant k, \quad k_1 \geqslant 0, \quad k_2 > 0. \tag{1b}$$

The result is derived by artificially increasing the system order by one and applying a Lyapunov function to the resulting equivalent system. The manner in which the sector and slope informations of the nonlinearity are accounted for is very much distinct from that in the existing approaches.

II. Main Result

Theorem: For the null solution of (1) to be ASIL, it is sufficient that there exists a real number $q \geqslant 0$ such that the following is satisfied:

$$\left[\frac{1}{k_2} + \left(1 - \frac{k_1}{k_2}\right)\operatorname{Re} G(z) - k_1 |G(z)|^2\right]\operatorname{Re}(1-z)$$
$$+ q\left[\frac{1}{k} + \operatorname{Re} G(z)\right] > 0, \qquad \text{for all } |z| = 1. \tag{2}$$

Proof: The state-space representation of the system is

$$\begin{aligned}
x_1(r+1) &= x_2(r)\\
x_2(r+1) &= x_3(r)\\
&\vdots\\
x_{n-1}(r+1) &= x_n(r)\\
x_n(r+1) &= -a_1 x_1(r) - a_2 x_2(r) \cdots - a_n x_n(r)\\
&\quad - f(h_1 x_1(r) + h_2 x_2(r) + \cdots + h_n x_n(r)).
\end{aligned} \tag{3}$$

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