Synthesis of Fuzzy Model-Based Designs to Synchronization and Secure Communications for Chaotic Systems

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***Abstract*—This paper presents synthesis approaches for synchronization and secure communications of chaotic systems by using fuzzy model-based design methods. Many well-known continuous and discrete chaotic systems can be exactly represented by T–S fuzzy models with only one premise variable. According to the applications on synchronization and signal modulation, the general fuzzy models may have either i) common bias terms; or ii) the same premise variable and driving signal. Then we propose two types of driving signals, namely, fuzzy driving signal and crisp driving signal, to deal with the asymptotical synchronization and secure communication problems for cases i) and ii), respectively. Based on these driving signals, the solutions are found by solving LMI problems. It is worthy to note that many well-known chaotic systems, such as Duffing system, Chua's circuit, Rössler's system, Lorenz system, Henon map, and Lozi map can achieve their applications on asymptotical synchronization and recovering messages in secure communication by using either the fuzzy driving signal or the crisp driving signal. Finally, several numerical simulations are shown to verify the results.**

***Index Terms*—Chaotic synchronization, linear matrix inequality, T–S fuzzy models.**

I. Introduction

In RECENT years, fuzzy systems have been applied to identification and control of nonlinear systems. Indeed, when a fuzzy representation of a nonlinear system is described by IF–THEN rules, the control problem then becomes to find a local linear/nonlinear compensator to achieve the desired objective. Many researches on this issue are carried out based on Takagi–Sugeno (T–S) fuzzy models [1]–[3], where the consequent parts represent local linear models. The controller and observer designs were proceeded by using the parallel distributed compensation concept. Then the stability of the overall system is related to finding a common symmetric positive definite matrix from linear matrix inequalities (LMI's) problem [4]. As pointed out in [5]–[9], the benefit of using a fuzzy model-based design is straightforward to obtain a controller or an observer.

The pioneering work of Carroll and Pecora [10], [11] has led to many works regarding synchronization of two chaotic systems [11]–[13]. According to the synthesis method proposed in [11], chaotic synchronization is where two chaotic systems with suitable coupling produce identical oscillations. Chaotic dynamics are deterministic but extremely sensitive to initial conditions. Even infinitesimal changes in initial condition will lead to an exponential divergence of orbits. The problem of chaotic synchronization is defined that given different initial conditions between drive and response systems, it is possible to find a method to make the states of drive and response system to achieve synchrony. Many theories [12]–[15] have been proposed to achieve the synchronized manner from master–slave configuration. This master–slave configuration consists of the original chaotic system as a *drive system* to provide a *driving signal* to drive another system called the *response system* to synchrony. Several control approaches, including model reference control and observer design, are widely used for synchronization.

Chaotic signals are typically broadband, noiselike, and difficult to predict, they can be used in various context for masking information-bearing waveforms. They can also be used as modulating waveforms in spread spectrum systems. This property leads to some interesting communications applications. For example, the chaotic signal masking technique introduced in [16] appears to be a potentially useful approach to secure communications. In a second approach to secure communications, the information-bearing waveform is used to modulate a transmitter coefficient. The corresponding synchronization error in the receiver can then be used to detect binary-valued bit stream. For chaotic communications, the receiver is driven by a scalar coupling channel from the transmitter. At the transmitter, the idea of chaotic masking [16], [17] is to directly add the message in a noise-like chaotic signal in a secure manner, while chaotic modulation [18]–[21] is by injecting the message into a chaotic system as spread-spectrum transmission. Later, at the receiver, a coherent detector and some signal processing is thus employed to recover the message from the received signal. These approaches have been developed as an application for chaotic synchronization. However, the methods for chaotic synchronization and secure communications have limitations. Most schemes must use high gains in designed parameters from assuming Lipschitz conditions of nonlinear terms [14], [18], or transmitting the nonlinear terms [15], such that the system noises are also amplified in the system loop. Recently to overcome these drawbacks, the control and synchronization chaotic systems using the T–S fuzzy modeling and their stability analysis have been investigated extensively [22]–[24]. In [23], Tanaka *et al.* proposed a fuzzy feedback law to deal

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with the synchronization and model following control for chaotic systems. In their work, the feedback law is realized via exact linearization (EL) techniques and by solving LMI problems. Although the EL techniques are developed such that the stability is ensured, the scheme is no longer suitable to secure communications due to the effects of signal masking and modulation. Moreover, since the method in [23] is developed from the controller point of view, it may require transmitting full states in dealing with synchronization problems.

In light of the fact that synchronization issues are closely related to the observer design, two methods are proposed in this paper to solve synchronization and secure communication with a scalar signal from an observer point of view. We first introduce how to present chaotic systems by T–S fuzzy models. The proposed method of building T–S fuzzy model is applicable (for which we have verified) to following chaotic systems: In discrete-time *Lure type* chaotic systems i) logistic and parabolic map in one-dimensional system; ii) Henon, Lozi, cubic map in two-dimensional system; iii) some of the three-dimensional systems in G. Baier *et al.* [26]; and iv) for higher order system the generalized Henon map. In continuous time chaotic systems i) Chua's circuit; ii) Lorenz system iii) Duffing and Van Der Pol oscillator; and iv) Rössler and transformed Rössler system [12]. All of the well-known continuous and discrete chaotic systems can be exactly represented by T–S fuzzy models with only one premise variable. In addition, most systems may have common bias terms in the fuzzy models. For models with a common bias term, a fuzzy driving signal can be adopted to achieve synchronization on two chaotic systems or to mask the message in secure communications. When some LMI conditions are held, the design parameters exist and can be found. On the other hand, without restricting common bias terms in the fuzzy model, typical (crisp) driving signals are employed. Here, the crisp driving signal is chosen same as the premise variable of the corresponding fuzzy chaotic model. In this case, the LMI's due to EL conditions can be removed. Although the latter approach is always simple in its design procedure, it is interesting to note that some chaotic systems, e.g. Rössler's system, does not suit this approach. For Rössler's system, we cannot find a solution from its LMI conditions when the crisp driving signal is applied. However, we observe that all the well-known chaotic systems can be applied to synchronization and secure communications either by fuzzy driving signal or by crisp driving signal. Notice that the T–S fuzzy model can exactly represent the chaotic systems. Meanwhile, we introduce the chaotic modulation as the communication structure. Hence the existence of the solution for the LMI conditions theoretically implies that the message can be perfectly recovered.

The rest of the paper is organized as follows: In Sections II and III, we establish a T–S fuzzy model which can exactly represent chaotic systems. Then, fuzzy synchronization and secure communications for continuous-time and discrete-time systems are investigated in Sections IV and V, respectively. Two approaches, namely, fuzzy driving signal and crisp driving signal are then introduced. The design parameters are presented by solving LMI conditions. In Section VI, numerical simulations are carried out on typical chaotic systems using the proposed method. Finally, some conclusions are made in Section VII.

II. Takagi–Sugeno Fuzzy Model

The T–S fuzzy dynamic model, which originates from Takagi and Sugeno [1], is described by fuzzy IF–THEN rules in which the consequent parts represent local linear models. In this section, we propose a systematic methodology of exactly presenting nonlinear systems by T–S fuzzy models. The methodology can yield many T–S fuzzy representations. Then a compact fuzzy model can be obtained by a careful selection of rule number and parameters. Consider a general nonlinear dynamic equation as follows:

$$sx(t) = f(x(t)) + g(x(t))u(t) \tag{1}$$

where $sx(t)$ are $\dot{x}(t)$ and $x(t+1)$ in continuous-time and discrete-time systems, respectively; $x \in R^n$, and $u \in R^p$ are the state and control input vectors, respectively; and $f(\cdot)$ and $g(\cdot)$ are nonlinear functions with appropriate dimensions.Then the fuzzy model is composed of the following rules:

$Plant\ Rule\ i$:
IF $z_1(t)$ is F_{1i} and $\cdots$ and $z_g(t)$ is F_{gi} THEN
$$sx(t) = A_i x(t) + B_i u(t) + \eta_i(t), \qquad i = 1, 2, \cdots, r \tag{2}$$

where $z_1(t) \sim z_g(t)$ are the premise variables which would consist of the states of the system; $F_{ji}(j = 1, 2, \cdots, g)$ are fuzzy sets; r is the number of fuzzy rules; A_i and B_i are system matrices with appropriate dimensions; $\eta_i(t)$bias term which is generated by the exact fuzzy modeling procedure. The continuous and discrete-time fuzzy systems are denoted as CFS and DFS, respectively.

Using the singleton fuzzifier, product fuzzy inference and weighted average defuzzifier, the final outputs of the fuzzy systems are inferred as follows:

$$sx(t) = \sum_{i=1}^{r} \mu_i(z(t))\{A_i x(t) + B_i u(t) + \eta_i(t)\} \tag{3}$$

where $z(t) = [z_1(t)\ \ z_2(t)\ \ \cdots\ \ z_g(t)]^T$, and $\mu_i(z(t)) = (\omega_i(z(t)))/(\sum_{i=1}^{r} \omega_i(z(t)))$ with $\omega_i(z(t)) = \prod_{j=1}^{g} F_{ji}(z_j(t))$. Note that $\sum_{i=1}^{r} \mu_i(z(t)) = 1$ for all t, where $\mu_i(z(t)) \geq 0$, for $i = 1, 2, \cdots, r$, are regarded as the normalized weights.

Now, focus on constructing a T–S fuzzy model (2) which exactly represents the nonlinear system (1). The vector function $f(x) + g(x)u(t)$ is expressed as fuzzy inferred outputs $\sum_{i=1}^{r} \mu_i(z)\{A_i x(t) + B_i u(t) + \eta_i(t)\}$ in (3). This means that when we specify the fuzzy membership functions in premise parts and associated entries of matrices A_i, B_i, and η_i in the consequence parts, the nonlinear system (1) may be represented by a T–S fuzzy model. To this end, the consistence of the nonlinear term in the system and its associated fuzzy representation are emphasized here. Without loss of generality, fuzzy modeling methods are proposed for three cases of nonlinear terms, that is A) only one variable in a nonlinear term; B) multi-variables in a nonlinear term; and C) multiple nonlinear terms in a system.

It is noted that the fuzzy systems would use the singleton fuzzifier, product fuzzy inference, and weighted average defuzzifier in this paper. The fuzzy modeling is only interesting the region of the system trajectory in the set $\Omega \equiv \{x(t) \in R^n | \|x(t)\| \le \delta\}$ for some δ. For some systems, such as chaotic systems, the existence of parameter δ is natural.

Case A—Only One Variable in a Nonlinear Term: Here, we intend to specify the membership functions and the associated coefficients in consequent parts such that a nonlinear term can be represented by a fuzzy system. Consider a single scalar nonlinear function $f(x_k)$ which depends only on one state variable x_k. Let the nonlinear term $f(x_k)$ take the form $\phi(x_k)x_m$, where

$$x_m = \begin{cases} x_k, & \text{if } \lim_{\substack{x \in \Omega \\ x_k \to 0}} \dfrac{f(x_k)}{x_k} \in L_\infty \\ 1, & \text{otherwise} \end{cases}$$

then the function $\phi(x_k)$ is well defined. Take x_k which forms the function $\phi(x_k)$ as the premise variable, then the fuzzy representation is composed of the following fuzzy rules:

$$\begin{aligned} &Rule\ i\text{: IF } x_k \text{ is } F_i \text{ THEN} \\ &\qquad \hat{f} = d_i x_m,\ i = 1, 2, \cdots, r \end{aligned}$$

where F_i is the fuzzy set, $\hat{f}$ is a fuzzy representation of $f(x_k)$;, and d_i is a constant coefficient to be determined.The fuzzy inferred output is written as

$$\hat{f}(x_k) = \frac{\sum_{i=1}^{r} \omega_i(x_k) d_i}{\sum_{i=1}^{r} \omega_i(x_k)} x_m$$

with $\omega_i(x_k) = F_i(x_k)$, which must equal to $\phi(x_k)x_m$. Without loss of generality, it is required that $\sum_{i=1}^r \omega_i(x_k) = 1$, which further yields $\phi(x_k) = \sum_{i=1}^r \omega_i(x_k) d_i$. Thus $f(x_k)$ can be exactly represented by a fuzzy system by suitably assigning $F_i(x_k)$ and d_i. Note that in this setting, the other linear terms, for instance θx_l is with the consequent part: $\hat{f} = \theta x_l$. Then the inferred output is

$$\hat{f} = \frac{\sum_{i=1}^{r} \omega_i(x_k) \theta x_l}{\sum_{i=1}^{r} \omega_i(x_k)}$$

which exactly equals θx_l. For demonstration, we let $r = 2$ and specify the membership functions. From $\omega_1 + \omega_2 = 1$ and $d_1 \omega_1 + d_2 \omega_2 = \phi(x_k)$, we have

$$\omega_1 = \frac{-d_2}{d_1 - d_2} + \frac{1}{d_1 - d_2} \phi(x_k), \qquad \omega_2 = 1 - \omega_1.$$

Care must be taken to determine the value of d_1 and d_2 such that $\omega_i(x_k) \in [0\ 1]$ for all $x \in \Omega$. For instance let $d_1 = -d_2 = d$ in which d is the upper bound of $\phi(x_k)$, i.e., $d = \sup_{x \in \Omega} |\phi(x_k)|$. This results in $F_1(x_k) = (1/2)(1 + (1/d)\phi(x_k))$ and $F_2(x_k) = (1/2)(1 - (1/d)\phi(x_k))$. Also, it is reasonable that when $\phi(x_k) \ge 0$ for all $x \in \Omega$, the fuzzy sets can be chosen as

$$F_1(x_k) = \frac{1}{d} \phi(x_k), \qquad F_2(x_k) = 1 - \frac{1}{d} \phi(x_k)$$

with $d_1 = d$ and $d_2 = 0$. Accordingly, the classified membership functions of the variable x_k are usually chosen with the sum of 1 for simplification.

Remark 1: We can conclude that this T–S fuzzy modeling approach requires i) $\omega_i(x_k) = F_i(x_k) \ge 0$; ii) $\sum_{i=1}^r \omega_i(x_k) = 1$ at $x(t) \in \Omega$; and check whether iii) $\lim_{\substack{x \in \Omega \\ x_k \to 0}} (f_1(x_k)/x_k) \in L_\infty$. If condition iii) is not satisfied, then the bias term will be yielded. In light of this, most nonlinear systems can be represented as T–S fuzzy models. The main problem is that the fuzzy rules will increase drastically as the nonlinear term becomes more complex.

Case B—Multi-Variables in a Nonlinear Term: A complex system usually has nonlinear terms which depend on more than one variable. Here consider a single scalar nonlinear function $f(x)$ in which $x = [x_1\ x_2\ \cdots\ x_n]^T$, where many state variables appear in it. Assume that the nonlinear term $f(x)$ can be expressed as $f(x) = \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)x_m$, where

$$x_m = \begin{cases} x_k, & \text{if } \lim_{\substack{x \in \Omega \\ x_k \to 0}} \dfrac{f(x)}{x_k} \in L_\infty \\ & \text{for some } k = 1, 2, \cdots, n \\ 1, & \text{otherwise} \end{cases} \tag{4}$$

then the function $\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)$ is well defined.

Let variable x_k in $\phi_k(\cdot)$, for $k = 1, 2, \cdots, n$, as premise variables, then the ith rule of the fuzzy system is of the following form:

$$\begin{aligned} &Rule\ i\text{: IF } x_1 \text{ is } F_{1i} \text{ and } \cdots \text{ and } x_n \text{ is } F_{ni} \text{ THEN} \\ &\qquad \hat{f} = d_i x_m, \qquad i = 1, 2, \cdots, r \end{aligned} \tag{5}$$

where $F_{ki}(k = 1, 2, \cdots, n)$ is the fuzzy set, $\hat{f}$ is a fuzzy representation of $f(x)$, and $d_i$is to be determined later. The final output of the fuzzy system is inferred as follows:

$$\hat{f} = \frac{\sum_{i=1}^{r} \omega_i(x) d_i}{\sum_{i=1}^{r} \omega_i(x)} x_m$$

where $\omega_i(x) = \prod_{k=1}^n F_{ki}(x_k)$, and $F_{ki}(x_k)$ is the grade of membership of x_k in F_{ki}. Inspired by Case A, let $\sum_{i=1}^r \omega_i(x) = 1$, and the sum of the grade for all classified membership functions for each variable x_k is equal to 1. Therefore, the membership functions of x_k and coefficient d_i would be chosen such that $F_{ki}(x_k) \in [0\ 1]$, $\sum_{i=1}^r \omega_i(x) = 1$, and $\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n) = \sum_{i=1}^r \omega_i(x) d_i$.

To illustrate the modeling scheme proposed herein, a nonlinear function $f(x_k, x_l)$ is considered and will be expressed as $\phi_k(x_k)\phi_l(x_l)x_m$. According to the above discussion, we need $\sum_{i=1}^r \omega_i(x_k, x_l) = 1$, and $\sum_{i=1}^r \omega_i(x_k, x_l)d_i = \phi_k(x_k)\phi_l(x_l)$. For simplification, let $r = 4$ and $\{F_{ka}, F_{kb}\}$, $\{F_{la}, F_{lb}\}$ be the classified fuzzy sets of variables x_k and x_l, respectively. The grades satisfy $F_{ka}(x_k) + F_{kb}(x_k) = 1$, and $F_{la}(x_l) + F_{lb}(x_l) = 1$. If the fuzzy rules are chosen with $F_{k1} = F_{k3} = F_{ka}$, $F_{k2} = F_{k4} = F_{kb}$, $F_{l1} = F_{l2} = F_{la}$,

$F_{l3} = F_{l4} = F_{lb}$, and $d_1 = -d_2 = -d_3 = d_4 = d_k d_l$, then this yields

$$F_{ka}(x_k) = \frac{1}{2}\left(1 + \frac{\phi_k(x_k)}{d_k}\right), \quad F_{kb}(x_k) = 1 - F_{ka}(x_k)$$
$$F_{la}(x_l) = \frac{1}{2}\left(1 + \frac{\phi_l(x_l)}{d_l}\right), \quad F_{lb}(x_l) = 1 - F_{la}(x_l)$$

where d_k and d_l are the upper bounds of ϕ_k and ϕ_l, respectively.

Notice that if the nonlinear function $f(x)$ can not be expressed as $\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)x_m$, then the nonlinear term can not be exactly represented in a fuzzy system by this method. In addition, if a nonlinear function $g(x)$ satisfies $g(x) = \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)$, then $g(x)u$ can be derived in a fuzzy representation by directly setting $x_m = u$ and yielding the fuzzy rule as (5).

Case C—Multiple Nonlinear Terms in a System: By introducing the fuzzy modeling methods in Cases A & B, more than one nonlinear term would be simultaneously considered in a system. When a nonlinear $h \times 1$ vector $f(x) = [f_1(x)\ f_2(x)\ \cdots\ f_h(x)]^T$ is considered, each element of $f(x)$ is assumed to satisfy $f_q(x) = \phi_{1q}(x_1)\phi_{2q}(x_2)\cdots\phi_{nq}(x_n)x_{mq}$ $(q = 1, 2, \cdots, h)$, where x_{mq} is defined similar to (4). Then $\phi_{1q}(x_1)\phi_{2q}(x_2)\cdots\phi_{nq}(x_n)$ is well defined. According to Cases A & B, the fuzzy system presenting the nonlinear terms are described as

$$\begin{aligned}&\textit{Rule } i\text{: IF } x_1 \text{ is } F_{1i} \text{ and } \cdots \text{ and } x_n \text{ is } F_{ni} \text{ THEN}\\ &\qquad \hat{f} = [\,d_{i1}x_{m1} \quad d_{i2}x_{m2} \quad \cdots \quad d_{ih}x_{mh}\,]^T\\ &\qquad i = 1, 2, \cdots, r\end{aligned} \tag{6}$$

which has the inferred output as shown in the equation at the bottom of the page, with $\omega_i(x) = \prod_{k=1}^{n} F_{ki}(x_k)$. The remaining procedure is same as Case B. It is noted that if the nonlinear terms $f_q(x)$, for $q = 1, 2, \cdots, h$, have the common factor, then the number of fuzzy rules may be reduced. Therefore the fuzzy system (6), accompanied with the fuzzy modeling for linear parts, provides a general method to represent nonlinear system (1) by the T–S fuzzy model (2).

III. Fuzzy Modeling of Chaotic Systems

To realize a fuzzy model-based design, chaotic systems should first be exactly represented by T–S fuzzy models. From the investigation of many well-known continuous-time and discrete-time chaotic systems mentioned in Introduction, we found that nonlinear terms have a common variable or depend only on one variable. If we take it as the premise variable of fuzzy rules, a simple fuzzy dynamic model can be obtained and will exactly represent chaotic systems in their naturally existed region Ω. Since chaotic systems do not have control inputs, $B_i = 0$ for all i in (2) for the modeling addressed below.

TABLE I
Different Driving Signal Schemes for Various Chaotic Systems

Chaotic Systems		Synchronization: Fuzzy Driving Signal	Synchronization: Crisp Driving Signal
Common Bias Term	Duffing	Yes	Yes
	Chua	Yes	Yes
	Rossler	Yes	No Solution for LMIs
	Lorenz	No Solution for LMIs	Yes
	Henon Map	Yes	Yes
Noncommon Bias Term	Transformed Rossler	Not Applicable	Yes
	Lozi Map	Not Applicable	Yes

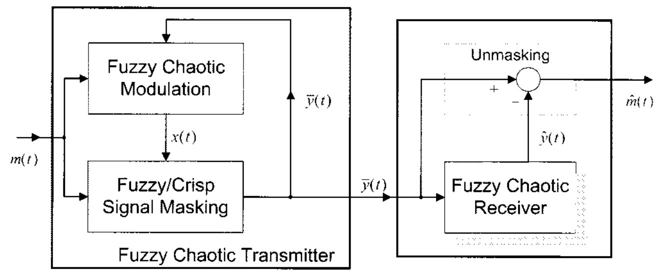


Fig. 1. Secure communication block diagram using fuzzy/crisp coupling signal.

The continuous-time chaotic systems [12] to be exactly represented by T–S fuzzy models will be considered in the following:

Duffing System (Only One Variable in the Nonlinear Term):

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= 1.1x_1(t) - x_1^3(t) - 0.4x_2(t) + 1.8\cos(1.8t).\end{aligned}$$

It is clear that the Duffing system has a nonlinear term $x_1^3(t)$ satisfying $x_1^3(t) = \phi(x_1(t))x_1(t)$ with $\phi(x_1(t)) = x_1^2(t)$. Thus the premise variable is chosen as $x_1(t)$. In the region of interest, the fuzzy model which exactly represents the Duffing system is as follows:

$$\begin{aligned}&\textit{Rule } i\text{: IF } x_1(t) \text{ is } F_i \text{ THEN}\\ &\qquad \dot{x}(t) = A_i x(t) + B_i u(t) + \eta_i(t), \qquad i = 1, 2\end{aligned}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1.1 - d & -0.4 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ 1.1 & -0.4 \end{bmatrix}$$
$$\eta_1(t) = \eta_2(t) = \begin{bmatrix} 0 \\ 1.8\cos(1.8t) \end{bmatrix}$$

$$\hat{f} = \begin{bmatrix} \dfrac{\sum_{i=1}^{r}\omega_i(x)d_{i1}}{\sum_{i=1}^{r}\omega_i(x)}x_{m1} & \dfrac{\sum_{i=1}^{r}\omega_i(x)d_{i2}}{\sum_{i=1}^{r}\omega_i(x)}x_{m2} & \cdots & \dfrac{\sum_{i=1}^{r}\omega_i(x)d_{ih}}{\sum_{i=1}^{r}\omega_i(x)}x_{mh} \end{bmatrix}^T$$

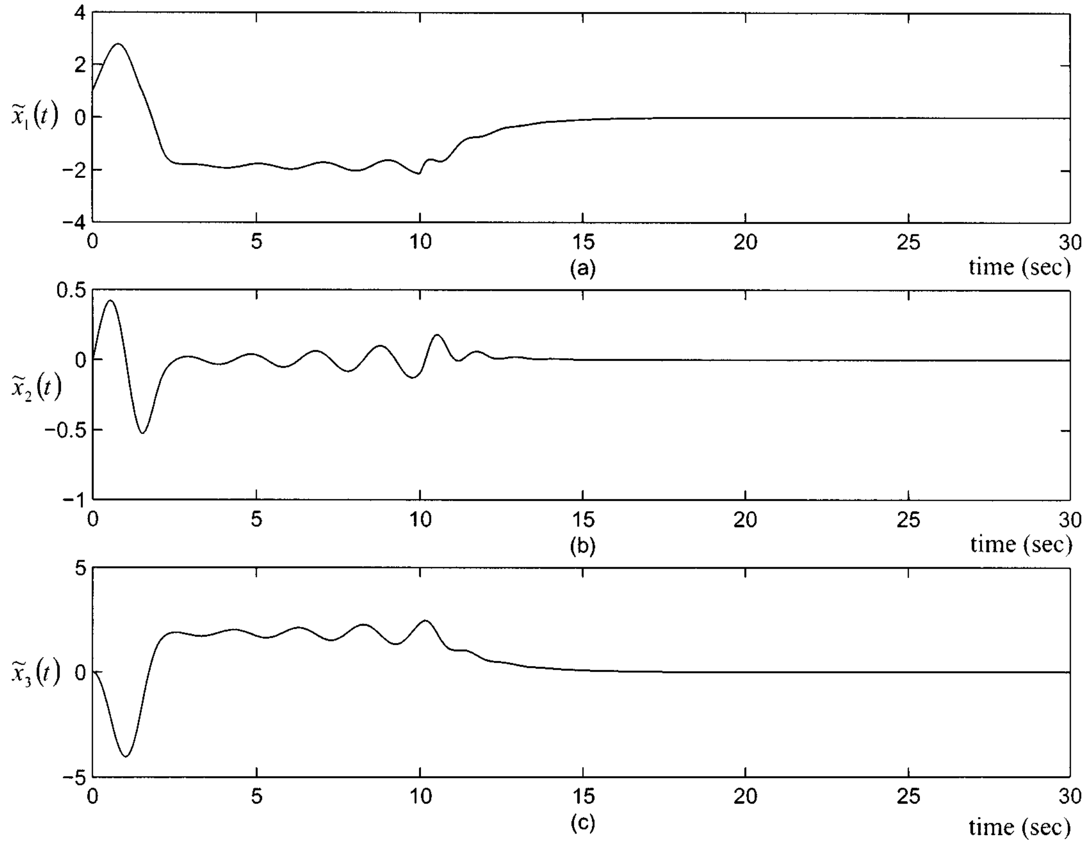


Fig. 2. (a) Synchronization error $\tilde{x}_1(t)$; (b) synchronization error $\tilde{x}_2(t)$; (c) synchronization error $\tilde{x}_3(t)$ of Chua's circuit with fuzzy driving signal activated at $t \geq 10$.

and the fuzzy sets are $F_1(x_1(t)) = (x_1^2(t)/d)$, $F_2(x_1(t)) = 1-(x_1^2(t)/d)$, where $d = \sup_{x\in\Omega} |\phi(x_1(t))| = 3$.

Chua's Circuit (Only One Variable in the Nonlinear Term):

$$\begin{aligned}\dot{x}_1(t) &= \sigma_1(-x_1(t)+x_2(t)-f(x_1(t)))\\ \dot{x}_2(t) &= x_1(t)-x_2(t)+x_3(t)\\ \dot{x}_3(t) &= -\sigma_2 x_2(t)\end{aligned}$$

with a nonlinear resistor $f(x_1(t)) = g_b x_1(t) + 0.5(g_a - g_b)(|x_1(t)+1| - |x_1(t)-1|)$, where $\sigma_1 = 10$, $\sigma_2 = 14.87$, $g_a = -1.27$ and $g_b = -0.68$. The nonlinear term $f(x_1(t))$ satisfies $\lim_{\substack{x_1(t)\to 0\\ x\in\Omega}} (f(x_1(t))/x_{1(t)}) = g_a$. Therefore, $f(x_1(t))$ is taken as $\phi(x_1(t))x_1$ with

$$\phi(x_1(t)) \equiv \begin{cases} f(x_1(t))/x_1(t), & x_1(t) \neq 0\\ g_a, & x_1(t) = 0.\end{cases}$$

Let $x_1(t)$ as the premise variable and choose the fuzzy sets to be $F_1(x_1(t)) = (1/2)(1-(\phi(x_1(t))/d))$ and $F_2(x_1(t)) = 1-F_1(x_1(t))$ with $d = \sup_{x\in\Omega}|\phi(x_1)| = 3$. Then, the fuzzy model which exactly represents Chua's circuit has

$$A_1 = \begin{bmatrix} (d-1)\sigma_1 & \sigma_1 & 0\\ 1 & -1 & 1\\ 0 & -\sigma_2 & 0\end{bmatrix}$$
$$A_2 = \begin{bmatrix} -(d+1)\sigma_1 & \sigma_1 & 0\\ 1 & -1 & 1\\ 0 & -\sigma_2 & 0\end{bmatrix}, \quad \eta_1 = \eta_2 = 0.$$

Rössler's System (Multi-Variables in the Nonlinear Term):

$$\begin{aligned}\dot{u}(t) &= -v(t) - w(t)\\ \dot{v}(t) &= u + av(t)\\ \dot{w}(t) &= c + u(t)w(t) - bw(t)\end{aligned} \tag{7}$$

where $a = 0.2$, $b = 5$, and $c = 0.2$. The nonlinear term $u(t)w(t)$ can have the extracted variable as $u(t)$ or $w(t)$. Here we choose $\phi(u(t)) = u(t)$ and let $u(t)$ as the premise variable of fuzzy rules. Then the fuzzy dynamic model which exactly represents the Rössler's system is with

$$A_1 = \begin{bmatrix} 0 & -1 & -1\\ 1 & a & 0\\ 0 & 0 & d-b\end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & -1 & -1\\ 1 & a & 0\\ 0 & 0 & -d-b\end{bmatrix}, \quad \eta_1 = \eta_2 = \begin{bmatrix} 0\\ 0\\ c\end{bmatrix}^T$$

and $F_1(x_1(t)) = (1/2)(1+(u(t)/d))$, $F_2(x_1(t)) = (1/2)(1-(u(t)/d))$ with $d = 10.5$.

Lorenz's System (Multiple Nonlinear Terms with a Common Factor):

$$\begin{aligned}\dot{x}_1(t) &= -10x_1(t) + 10x_2(t)\\ \dot{x}_2(t) &= 28x_1(t) - x_2(t) - x_1(t)x_3(t)\\ \dot{x}_3(t) &= x_1(t)x_2(t) - \tfrac{8}{3}x_3(t).\end{aligned}$$

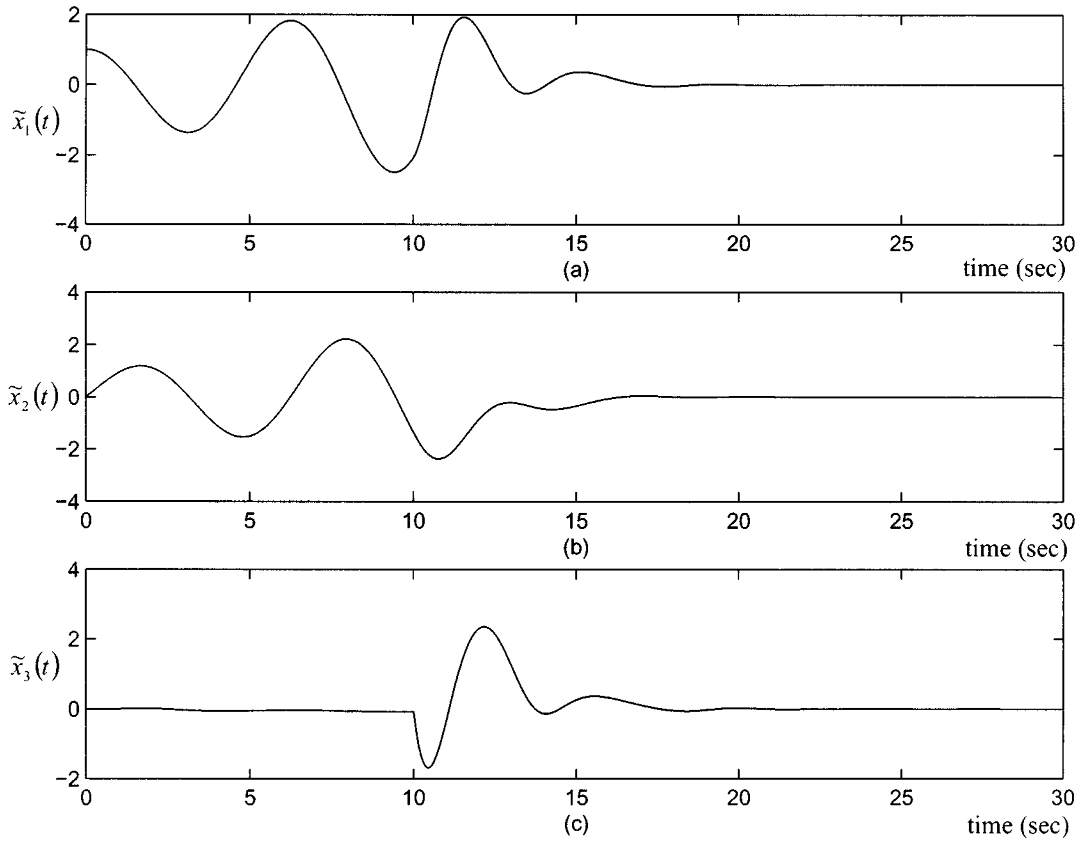


Fig. 3. (a) Synchronization error $\tilde{x}_1(t)$; (b) synchronization error $\tilde{x}_2(t)$; (c) synchronization error $\tilde{x}_3(t)$ of Rössler's system with fuzzy driving signal activated at $t \geq 10$.

The common factor of nonlinear terms $x_1(t)x_3(t)$ and $x_1(t)x_2(t)$ is $\phi(x_1(t)) = x_1(t)$. Therefore, the premise variable of fuzzy rules is chosen as $x_1(t)$, which satisfies $x_1(t) \in [-d\ d]$ with $d = 30$. The fuzzy model which exactly represents the Lorenz's system is:

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -d \\ 0 & d & -\frac{8}{3} \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & d \\ 0 & -d & -\frac{8}{3} \end{bmatrix}, \ \eta_1 = \eta_2 = 0$$

and the fuzzy sets are chosen as $F_1(x_1(t)) = (1/2)(1 + (x_1(t)/d))$, $F_2(x_1(t)) = (1/2)(1 - (x_1(t)/d))$.

Transformed Rössler's System (Multiple Nonlinear Terms in a System):

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) - \exp(x_3(t)) \\ \dot{x}_2(t) &= x_1(t) + ax_2(t) \\ \dot{x}_3(t) &= x_1(t) + c\ \exp(-x_3(t)) - b. \end{aligned}$$

For some applications, the Rössler's system (7) may be represented in other coordinates defined by $x_1 = u$, $x_2 = v$ and $x_3 = \ln(w)$ [18]. Since the nonlinear terms $\exp(x_3(t))$ and $\exp(-x_3(t))$ depend on the common variable $x_3(t)$, the premise variable is set as $x_3(t)$. However, the variable $x_3(t)$ of two nonlinear terms can not be extracted due to $\lim_{\substack{x \in \Omega \\ x_3 \to 0}} (\exp(x_3)/x_3) \notin L_\infty$ and $\lim_{\substack{x \in \Omega \\ x_3 \to 0}} (\exp(-x_3)/x_3) \notin L_\infty$. Thus bias terms will appear in the T–S fuzzy model. The transformed Rössler's system is exactly represented by the T–S fuzzy model with $r = 4$, and

$$A_i = \begin{bmatrix} 0 & -1 & 0 \\ 1 & a & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad i = 1, 2, 3, 4$$
$$\eta_1 = [-d \quad 0 \quad -b + cd]^T, \quad \eta_2 = [-d \quad 0 \quad -b - cd]^T$$
$$\eta_3 = [d \quad 0 \quad -b + cd]^T, \quad \eta_4 = [d \quad 0 \quad -b - cd]^T$$

with fuzzy sets $F_1(x_3(t)) = (1/2^2)(1 + (\exp(x_3(t))/d))(1 + (\exp(-x_3(t))/d))$, $F_2(x_3(t)) = (1/2^2)(1 + (\exp(x_3(t))/d))(1 - (\exp(-x_3(t))/d))$, $F_3(x_3(t)) = (1/2^2)(1 - (\exp(x_3(t))/d))(1 + (\exp(-x_3(t))/d))$, $F_4(x_3(t)) = (1/2^2)(1 - (\exp(x_3(t))/d))(1 - (\exp(-x_3(t))/d))$, and $d = 75$.

The discrete-time chaotic systems [12] to be exactly represented by T–S fuzzy models are as follows.

Henon Map (Only One Variable in a Nonlinear Term):

$$\begin{aligned} x_1(t+1) &= -x_1^2(t) + 0.3x_2(t) + 1.4 \\ x_2(t+1) &= x_1(t). \end{aligned}$$

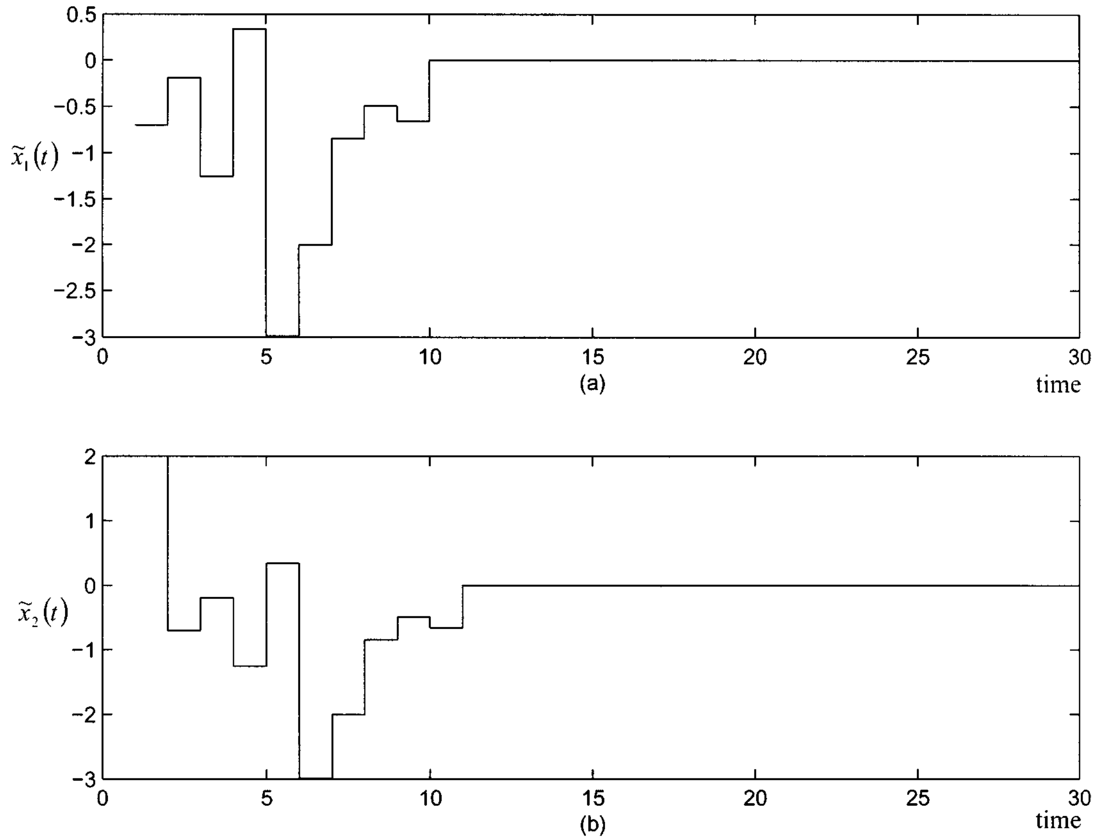


Fig. 4. (a) Synchronization error $\tilde{x}_1(t)$; (b) synchronization error $\tilde{x}_2(t)$ of Henon map with fuzzy driving signal activated at $t \geq 10$.

Since, the nonlinear term is $x_1^2(t)$, it follows that $\phi(x_1(t)) = x_1(t)$. Let $x_1(t)$ as the premise variable, then the equivalent fuzzy model can be constructed as

$$\begin{aligned}&\mathit{Rule}\ i\text{: IF } x_1(t) \text{ is } F_i \text{ THEN}\\ &\qquad x(t+1) = A_i x(t) + B_i u(t) + \eta_i(t), \qquad i = 1, 2\end{aligned}$$

where

$$A_1 = \begin{bmatrix} -d & 0.3 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} d & 0.3 \\ 1 & 0 \end{bmatrix}$$
$$\eta_1 = \eta_2 = \begin{bmatrix} 1.4 \\ 0 \end{bmatrix}$$

and the fuzzy sets are $F_1(x_1(t)) = (1/2)(1 + (x_1(t)/d))$, $F_2(x_1(t)) = (1/2)(1 - (x_1(t)/d))$ with $d = 2$.

Lozi Map (Only One Variable in the Nonlinear Term):

$$\begin{aligned} x_1(t+1) &= -1.8|x_1(t)| + x_2(t) + 3 \\ x_2(t+1) &= 0.25 x_1(t) \end{aligned}$$

which has the nonlinear term $|x_1(t)|$. Since $|x_1(t)|/x_1(t)$ is not well defined at $x_1(t) = 0$, let $\phi(x_1(t)) = |x_1(t)|$ and choose $x_1(t)$ as the premise variable of fuzzy rules. The equivalent fuzzy model can be constructed with

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix}, \quad \eta_1 = \begin{bmatrix} -1.8d + 3 \\ 0 \end{bmatrix}$$

$$\eta_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

The fuzzy sets are $F_1(x_1(t)) = (|x_1(t)|/d)$, $F_2(x_1(t)) = 1 - (|x_1(t)|/d)$, with $d = 3.5$.

According to these fuzzy representations, the general form of T–S fuzzy models for chaotic systems can be written as follows:

$$\begin{aligned}&\mathit{Rule}\ i\text{:}\quad \text{IF } z(t) \text{ is } F_i \text{ THEN}\\ &\qquad sx(t) = A_i x(t) + \eta_i(t), \qquad i = 1, 2, \cdots, r \end{aligned} \tag{8}$$

where the premise variable $z(t)$ is a proper state variable. From the observation of bias terms, many systems (except for the transformed Rössler's system and Lozi map) have common bias terms in fuzzy models, i.e., $\eta_i = \eta$, for $i = 1, 2, \cdots, r$. The following synchronization and secure communication of chaotic systems will be proposed based on the fuzzy dynamic model (8).

IV. Fuzzy Chaotic Synchronization Design

Based on the fuzzy modeling of chaotic systems in Section III, two approaches are proposed to achieve chaotic synchronization. The synchronization problem is to design the output of the drive system to force the response system to same internal states. According to chaotic fuzzy models, the design for two different driving signals are discussed, namely 1) synchronization with fuzzy driving signals; and 2) synchronization with crisp driving signals. The design procedures are developed as follows.

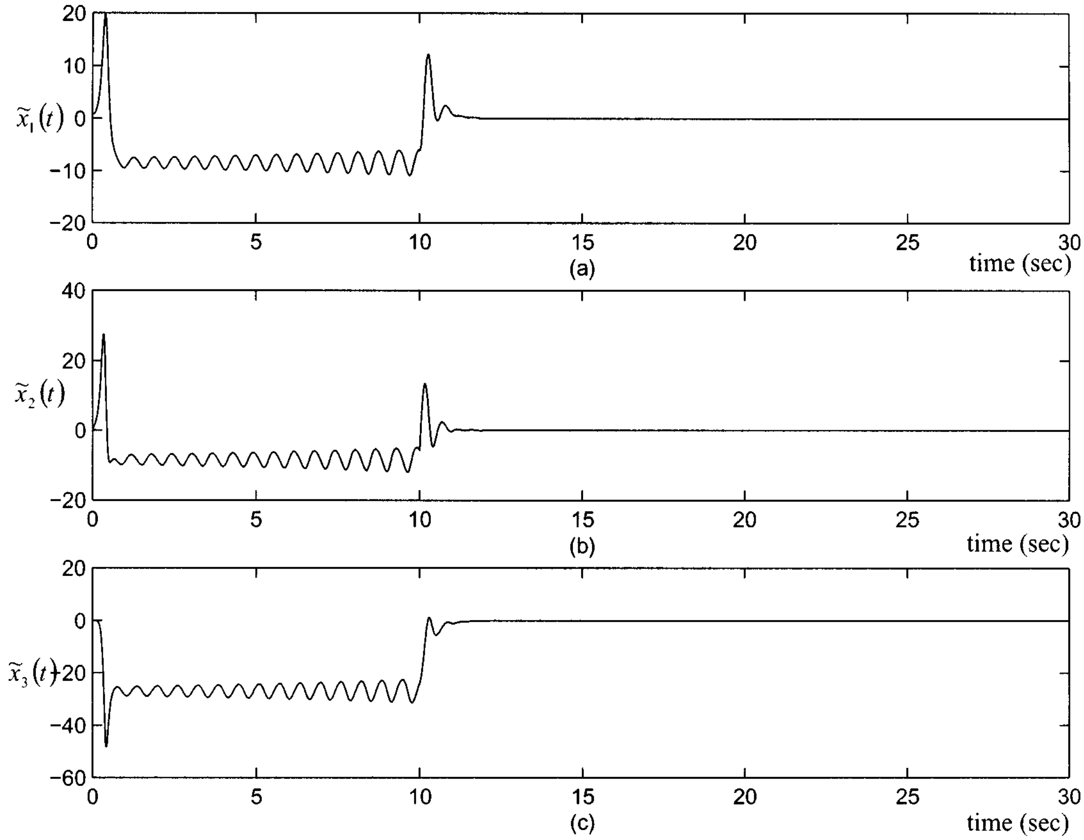


Fig. 5. (a) Synchronization error $\tilde{x}_1(t)$; (b) synchronization error $\tilde{x}_2(t)$; (c) synchronization error $\tilde{x}_3(t)$ of Lorenz's system with crisp driving signal activated at $t \geq 10$.

A. Synchronization with Fuzzy Driving Signals

Consider a class of chaotic systems which have common bias terms in fuzzy model representation, that is, the drive system is expressed as

Drive System Rule i:

$$\begin{aligned}&\text{IF } z(t) \text{ is } F_i \text{ THEN}\\ &sx(t) = A_i x(t) + \eta(t), \qquad i = 1, 2, \cdots, r.\end{aligned}$$

Then the fuzzy driving signal is generated by the following fuzzy rules:

Driving Signal Rule i:

$$\begin{aligned}&\text{IF } z(t) \text{ is } F_i \text{ THEN}\\ &y(t) = C_i x(t), \qquad i = 1, 2, \cdots, r\end{aligned}$$

where vectors C_i, for $i = 1, 2, \cdots, r$, are to be designed later. Then the overall inferred output of the drive system is:

$$sx(t) = \sum_{i=1}^{r} \mu_i(z(t))\{A_i x(t) + \eta(t)\} \tag{9}$$

$$y(t) = \sum_{i=1}^{r} \mu_i(z(t)) C_i x(t) \tag{10}$$

where $\mu_i(z(t)) = (\omega_i(z(t))/\sum_{i=1}^{r} \omega_i(z(t)))$ with $\omega_i(z(t)) = F_i(z(t)) \geq 0$. Using the fuzzy driving signal $y(t)$ as (10), the fuzzy response system is composed of the following rules:

Response System Rule i:

$$\begin{aligned}&\text{IF } \hat{z}(t) \text{ is } F_i \text{ THEN}\\ &s\hat{x}(t) = A_i\hat{x}(t) + \eta(t) + L(y(t) - \hat{y}(t))\\ &\hat{y}(t) = C_i\hat{x}(t), \qquad i = 1, 2, \cdots, r\end{aligned} \tag{11}$$

where the premise variable $\hat{z}(t)$ of the response system is the estimate of $z(t)$; $\hat{x}(t)$ presents the estimated state vector, and L is an appropriate $n \times 1$ vector. The overall inferred output of the response system is

$$\begin{aligned}s\hat{x}(t) &= \sum_{i=1}^{r} \mu_i(\hat{z}(t))\{A_i\hat{x}(t) + \eta(t) + L(y(t) - \hat{y}(t))\}\\ \hat{y}(t) &= \sum_{i=1}^{r} \mu_i(\hat{z}(t)) C_i \hat{x}(t)\end{aligned} \tag{12}$$

where $\mu_i(\hat{z}(t)) = \omega_i(\hat{z}(t))/\sum_{i=1}^{r} \omega_i(\hat{z}(t))$ with $\omega_i(\hat{z}(t)) = F_i(\hat{z}(t)) \geq 0$. Define error signal $\tilde{x}(t) = x(t) - \hat{x}(t)$. Then according to (9) and (12), the error dynamics of $\tilde{x}(t)$ can be expressed in

$$s\tilde{x}(t) = \sum_{i=1}^{r} \mu_i(z)(A_i - LC_i)x(t) - \sum_{i=1}^{r} \mu_i(\hat{z})(A_i - LC_i)\hat{x}(t). \tag{13}$$

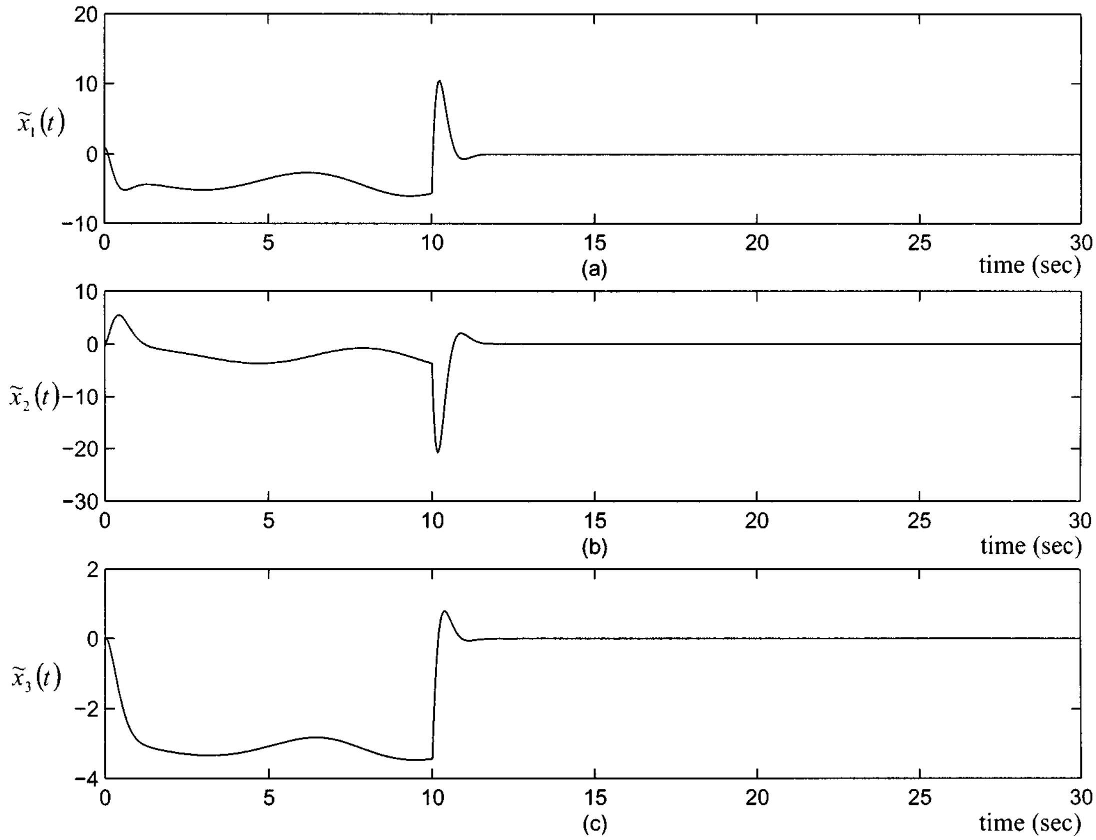


Fig. 6. (a) Synchronization error $\tilde{x}_1(t)$; (b) synchronization error $\tilde{x}_2(t)$; (c) synchronization error $\tilde{x}_3(t)$ of transformed Rössler's system with crisp driving signal activated at $t \geq 10$.

Theorem 1: The error dynamics (13) is exact linearization (EL) if given a vector L there exist gains C_i such that

$$\{(A_1 - LC_1) - (A_i - LC_i)\}^T \cdot \{(A_1 - LC_1) - (A_i - LC_i)\} = 0, \qquad 2 \leq i \leq r. \quad (14)$$

Then, the overall error dynamics is linearized as $s\tilde{x}(t) = G\tilde{x}(t)$, where $G = A_1 - LC_1 = A_i - LC_i$, for $2 \leq i \leq r$.

Proof: It is clear that if the condition of (14) is held then $G = A_1 - LC_1 = A_i - LC_i$, for $2 \leq i \leq r$. This implies the stability of the closed-loop system is reduced to analyze $s\tilde{x}(t) = G\tilde{x}(t)$. □

The following results for CFS and DFS are stated to ensure the stability of the overall system.

Theorem 2 (CFS): The error system described by (13) for CFS is uniformly asymptotically stable if there exist a common positive definite matrix P and gains C_i for $i = 1, 2, \cdots, r$, by solving the following eigenvalue problem (EVP) as shown in (15) and (16) at the bottom of the page, where $M_i = C_i P^{-1}$, for $i = 1, 2, \cdots, r$, and $X = P^{-1}$.

Proof: For the EL conditions (14), there exist a positive definite matrix X and a small constant $\varepsilon > 0$ such that

$$\varepsilon I - \{A_1X - LM_1 - (A_iX - LM_i)\}^T \cdot \{A_1X - LM_1 - (A_iX - LM_i)\} > 0, \qquad 2 \leq i \leq r.$$

This means if all elements in εX^{-2} are near zero in above inequalities for a proper choice of $\varepsilon > 0$, $X > 0$, i.e., $\varepsilon X^{-2} \approx 0$, then the EL conditions (14) are achieved. This implies that the error dynamics (13) can be expressed as $s\tilde{x}(t) = G\tilde{x}(t)$ once the inequalities (16) can be held. Therefore the error system with gains C_i should be designed to guarantee stability of the linearized error system (13). To this end, define a Lyapunov function candidate as $V(\tilde{x}(t)) = \tilde{x}^T(t)P\tilde{x}(t)$ with $P > 0$, and take

$$\underset{M_i,\, X}{\text{minimize}}\ \varepsilon$$

subject to $X > 0$, $\varepsilon > 0$

$$-A_iX - XA_i^T + M_i^T L^T + LM_i > 0, \text{ for all } i, \quad (15)$$

$$\begin{bmatrix} \varepsilon I & \{A_1X - LM_1 - (A_iX - LM_i)\}^T \\ \{A_1X - LM_1 - (A_iX - LM_i)\} & I \end{bmatrix} > 0, \qquad 2 \leq i \leq r \quad (16)$$

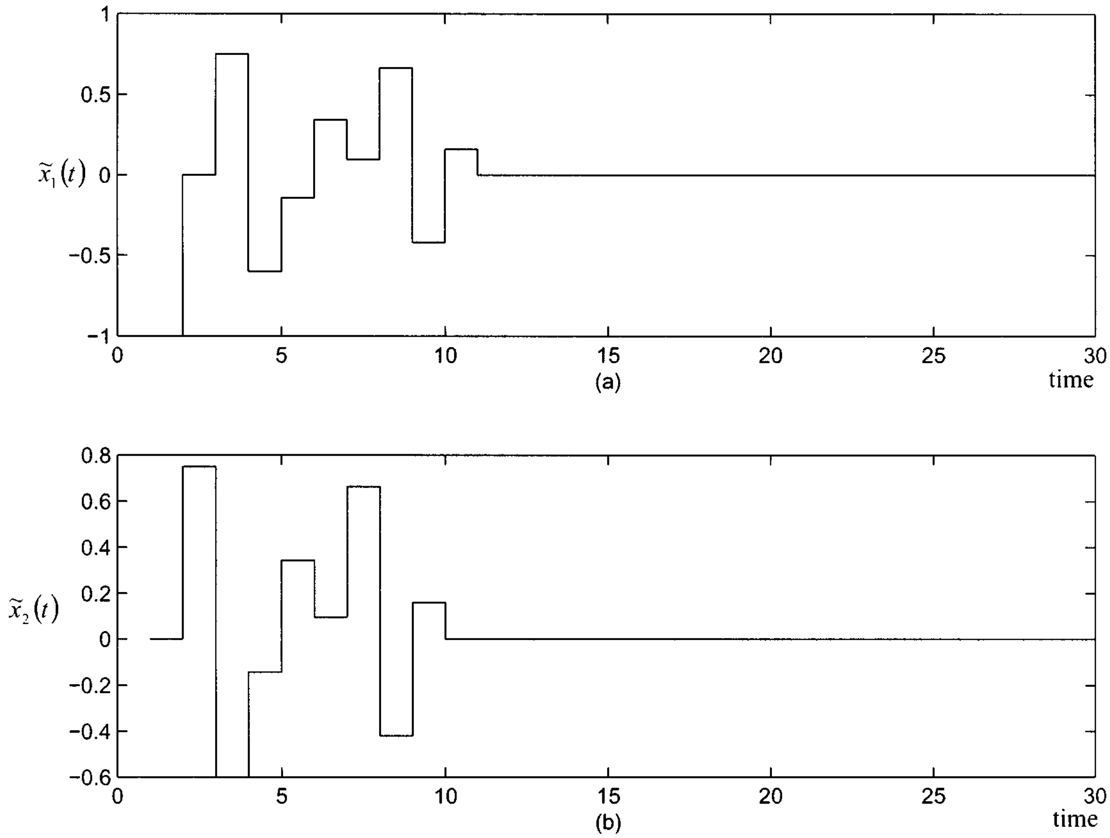


Fig. 7. (a) Synchronization error $\tilde{x}_1(t)$; (b) synchronization error $\tilde{x}_2(t)$ of Lozi map with crisp driving signal activated at $t \geq 10$.

the time derivative of $V(\tilde{x}(t))$ along the overall error dynamics. This yields

$$\dot{V}(\tilde{x}(t)) = \tilde{x}^T(G^T P + PG)\tilde{x}(t). \tag{17}$$

Thus, if the Riccati inequalities (15) are satisfied then $\dot{V}(\tilde{x}(t)) < 0$, which implies that error $\tilde{x}(t)$ asymptotically converges to zero as $t \to \infty$. □

Theorem 3 (DFS): The error system described by (13) for DFS is uniformly asymptotically stable if there exist a common positive definite matrix P and gains C_i, for $i = 1, 2, \cdots, r$, which can be determined by solving the following eigenvalue problem as shown in (18) and (19), shown at the bottom of the page where $M_i = C_i P^{-1}$, for $i = 1, 2, \cdots, r$, and $X = P^{-1}$.

Proof: The proof is similar to Theorem 2. If the solutions of the LMI design problem stated in (18) and (19) are feasible, the EL technique is realized by minimizing $\|\varepsilon X^{-2}\|$ near to zero. Moreover, there exists a Lyapunov function candidate for DFS as $V(\tilde{x}(t)) = \tilde{x}^T(t)P\tilde{x}(t) > 0$ with difference as

$$\triangle V(\tilde{x}(t)) = \tilde{x}(t)^T\{G^T PG - P\}\tilde{x}(t) \tag{20}$$

where $P > 0$. The conditions (18) ensure that $G^T PG - P < 0$, for $i = 1, 2, \cdots, r$, and $\triangle V(\tilde{x}(t)) < 0$. Then, the error dynamical system (13) has $\lim_{t\to\infty} \tilde{x}(t) = 0$. □

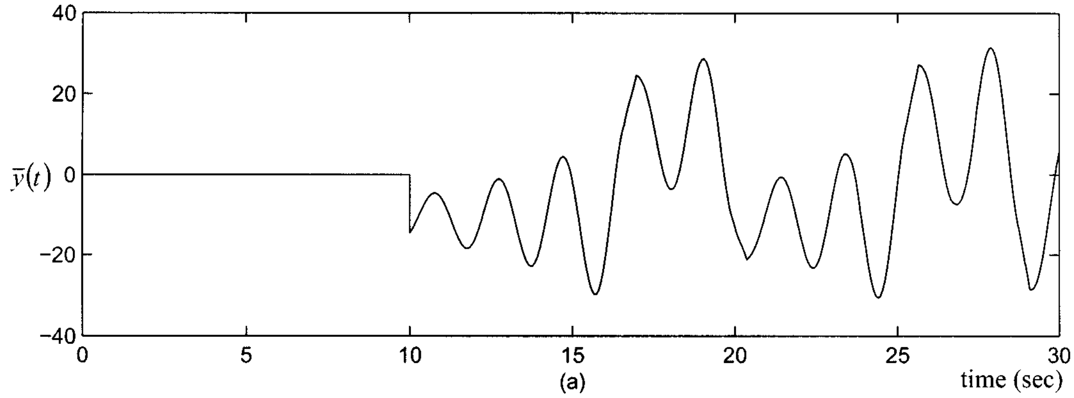
The EL technique similar in [23] plays a main role in this design scheme due the different premise variables between the drive and response systems. When the LMI's of EL conditions are held, the vectors C_i, for $i = 1, 2, \cdots, r$, are obtained by $C_i = M_i P$ from the solutions of X and M_i. However, the EL conditions are strict and complex. We will eliminate the conditions by designing other driving signals to achieve synchronization.

$$\underset{M_i, X}{\text{minimize}}\ \varepsilon$$

subject to $X > 0,\ \varepsilon > 0$

$$\begin{bmatrix} X & (A_i X - LM_i)^T \\ A_i X - LM_i & X \end{bmatrix} > 0 \text{ for all } i \tag{18}$$

$$\begin{bmatrix} \varepsilon I & \{A_1 X - LM_1 - (A_i X - LM_i)\}^T \\ A_1 X - LM_1 - (A_i X - LM_i) & I \end{bmatrix} > 0, \qquad 2 \leq i \leq r \tag{19}$$



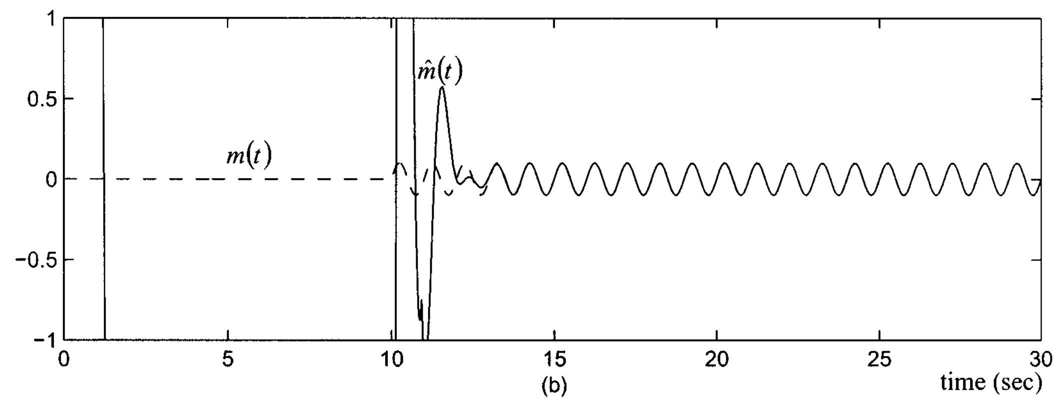


Fig. 8. (a) Coupling signal $\bar{y}(t)$; (b) original message $m(t)$ and recovered message $\hat{m}(t)$ using Chua's circuit with fuzzy signal masking activated at $t \geq 10$.

B. Synchronization with Crisp Driving Signals

In order to remove EL conditions in design procedure, the typical driving signal design is considered. Since the typical driving mechanism is designed by properly selecting a crisp output, the signal is called the crisp driving signal. Here, the crisp driving signal is chosen to be same as the premise variable of the fuzzy model for the corresponding chaotic systems, i.e., $y(t) = Cx(t) = z(t)$, and a known vector C. This means that the chaotic system (8) is taken as the drive system represented as

$Drive\ System\ Rule\ i$:

$$\text{IF } y(t) \text{ is } F_i \text{ THEN}$$
$$sx(t) = A_i x(t) + \eta_i(t), \qquad i = 1, 2, \cdots, r.$$

The overall inferred output can be written as

$$sx(t) = \sum_{i=1}^{r} \mu_i(y(t))\{A_i x(t) + \eta_i(t)\} \tag{21}$$
$$y(t) = Cx(t) \tag{22}$$

where $\mu_i(y(t)) = \omega_i(y(t))/\sum_{i=1}^{r} \omega_i(y(t))$ with $\omega_i(y(t)) = F_i(y(t)) \geq 0$. Therefore, it is straightforward to let the driving signal $y(t)$ as the premise variable of the response system. For synchronization, the response system is composed of the following rules:

$Response\ System\ Rule\ i$:

$$\text{IF } y(t) \text{ is } F_i \text{ THEN}$$
$$s\hat{x}(t) = A_i\hat{x}(t) + \eta_i(t) + L_i(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = C\hat{x}(t), \qquad i = 1, 2, \cdots, r \tag{23}$$

where L_i is a design gain determined later. The overall response system is inferred in the following

$$s\hat{x}(t) = \sum_{i=1}^{r} \mu_i(y(t))\{A_i x(t) + \eta_i(t) + L_i(y(t) - \hat{y}(t))\} \tag{24}$$
$$\hat{y}(t) = C\hat{x}(t). \tag{25}$$

Define error signal $\tilde{x}(t) = x(t) - \hat{x}(t)$. According to (21) and (24), the error dynamics of $\tilde{x}(t)$ is expressed as

$$s\tilde{x}(t) = \sum_{i=1}^{r} \mu_i(y(t))(A_i - L_i C)\tilde{x}(t). \tag{26}$$

The stability conditions for (26) is derived using Lyapunov method. Now, the main results will be addressed here.

Theorem 4 (CFS): The error system described by (26) for CFS is uniformly asymptotically stable if there exist a common positive definite matrix P and gains L_i, for $i = 1, 2, \cdots, r$, such that the following LMI's, with $N_i \equiv PL_i$,

$$-A_i^T P - PA_i + C^T N_i^T + N_i C > 0, \qquad \text{for all } i \tag{27}$$

have feasible solutions.

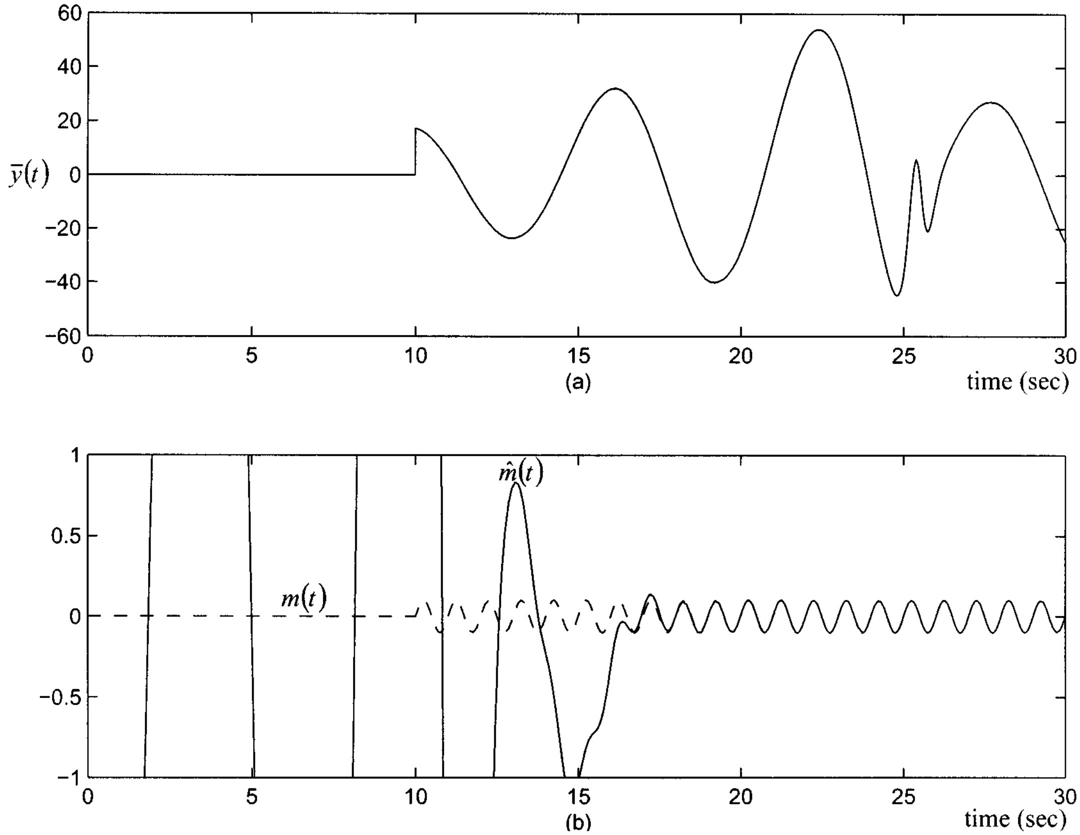


Fig. 9. (a) Coupling signal $\overline{y}(t)$; (b) original message $m(t)$ and recovered message $\hat{m}(t)$ using Rössler's system with fuzzy signal masking activated at $t \geq 10$.

Proof: Define the Lyapunov function candidate as $V(\tilde{x}(t)) = \tilde{x}^T(t)P\tilde{x}(t)$ with $P > 0$, then the time derivative of $V(\tilde{x}(t))$ along the error dynamics (26) is

$$\dot{V}(\tilde{x}(t)) = \sum_{i=1}^{r} \mu_i(y(t))\tilde{x}^T(t) \cdot \{(A_i - L_iC)^T P + P(A_i - L_iC)\}\tilde{x}(t). \quad (28)$$

Since (27) is satisfied, we denote the minimum positive definite matrix of the left hand side of (27) by Q. It follows that

$$\begin{aligned}\dot{V}(\tilde{x}(t)) &\leq -\sum_{i=1}^{r} \mu_i(y(t))\tilde{x}^T(t)Q\tilde{x}(t)\\ &= -\tilde{x}^T(t)Q\tilde{x}(t) < 0\end{aligned}$$

Hence $\tilde{x} = 0$ is uniformly asymptotically stable. □

Theorem 5 (DFS): The error system described by (26) for DFS is uniformly asymptotically stable if there exist a common positive definite matrix P and gains L_i, for $i = 1, 2, \cdots, r$, such that

$$\begin{bmatrix} P & (PA_i - N_iC)^T \\ PA_i - N_iC & P \end{bmatrix} > 0, \qquad \text{for all } i \quad (29)$$

where $N_i \equiv PL_i$.

Proof: The proof is similar as Theorem 4. Given a Lyapunov function candidate for DFS as $V(\tilde{x}(t)) = \tilde{x}^T(t)P\tilde{x}(t) > 0$, we have

$$\begin{aligned}\triangle V(\tilde{x}(t)) = &\sum_{i=1}^{r} \mu_i^2(y(t))\tilde{x}^T(t)[\overline{A}_i^T P\overline{A}_i - P]\tilde{x}(t)\\ &+ \sum_{i<j}^{r} \mu_i(y(t))\mu_j(y(t))\tilde{x}^T(t)\\ &\cdot [\overline{A}_i^T P\overline{A}_j + \overline{A}_j^T P\overline{A}_i - 2P]\tilde{x}(t) \quad (30)\end{aligned}$$

where $P > 0$, and $\overline{A}_i = A_i - L_iC$. Notice that if $\overline{A}_i^T P\overline{A}_i - P < 0$, then $\overline{A}_i^T P\overline{A}_j + \overline{A}_j^T P\overline{A}_i - 2P < 0$. This means if there are P and L_i such that the conditions (29) are held, then $\overline{A}_i^T P\overline{A}_i - P < 0$. Let $-Q$ denote the maximum negative definite matrix of $A_i^T P\overline{A}_i - P$ for all i. Then $\triangle V(\tilde{x}(t)) \leq -\tilde{x}^T(t)Q\tilde{x}(t) < 0$. Thus the synchronization error $\tilde{x}(t)$ asymptotically converges to zero as $t \to \infty$.

By solving LMI problems in (27) and (29), we can determine P and N_i. Then the gains L_i in Theorems 4 and 5 can be obtained from the relation $L_i = P^{-1}N_i$. Since the fuzzy models can exactly represent chaotic systems in any prescribed region Ω by selecting a proper sector parameter δ, the stability of two synchronization schemes is guaranteed in a semi-global region by the Lyapunov's direct method. In addition, after solving the corresponding LMI's design problems, the situations of applying two synchronization schemes to chaotic systems can be stated in Table I. According to this, chaotic systems in fuzzy models may have either i) common bias terms; or ii) the same premise

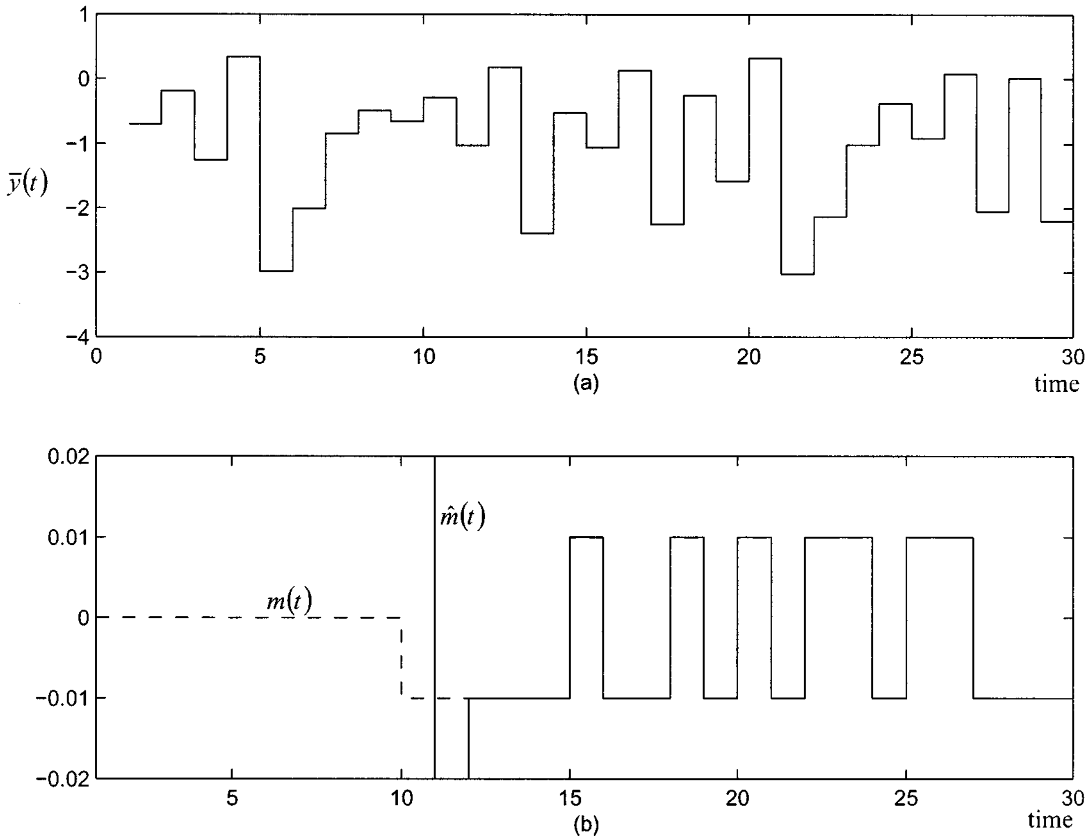


Fig. 10. (a) Coupling signal $\overline{y}(t)$; (b) original message $m(t)$ and recovered message $\hat{m}(t)$ using Henon map with fuzzy signal masking activated at $t \geq 10$.

variable and driving signal. Two cases can be solved by using fuzzy and crisp driving signals, respectively. In other words, all well-known chaotic systems discussed in Section III can achieve synchronization applications by using either the fuzzy driving signal or crisp driving signal or both. This means that fuzzy model-based synchronization is very flexible and useful in practical applications.

V. Fuzzy Modulated Chaotic Communications

In light of the T–S fuzzy modeling method proposed above, the chaotic modulation method of [19], [20] may be transformed into a fuzzy chaotic modulation architecture as shown in Fig. 1. From the diagram, we are able to observe that a message $m(t)$ is modulated by either fuzzy or crisp chaotic signal masking methods. This modulation process is carried out in the so-called fuzzy chaotic transmitter. Then the coupling signal $\overline{y}(t)$ is sent to the fuzzy chaotic receiver in which the message is extracted accordingly to different masking methods (fuzzy or crisp). Therefore, secure communications is achieved. The details of secure communications using the fuzzy modulation method is given in the following.

A. Fuzzy Signal Masking

Based on introducing fuzzy driving concept, a new scheme of modulated chaotic communication is proposed here. Inspired by previous works of modulated chaotic communication [18]–[21], the fuzzy modulated chaotic transmitter and the fuzzy signal masking are designed for a class of chaotic systems which have the common bias terms in fuzzy T–S models. Now, the fuzzy chaotic transmitter with message $m(t)$ embedded is given as

$Transmitter\ Rule\ i$:

IF $z(t)$ is F_i THEN

$$sx(t) = A_i x(t) + \eta(t) + Lm(t), \qquad i = 1, 2, \cdots, r$$

with the fuzzy masking mechanism:

$Masking\ Rule\ i$: IF $z(t)$ is F_i THEN

$$\overline{y}(t) = C_i x(t) + m(t), \qquad i = 1, 2, \cdots, r$$

where vector L is given, and C_i is to be designed later. The fuzzy inferred transmitter can be expressed in the form:

$$sx(t) = \sum_{i=1}^{r} \mu_i(z(t))\{G_i x(t) + \eta(t) + L\overline{y}(t)\} \tag{31}$$

$$\overline{y}(t) = \sum_{i=1}^{r} \mu_i(z(t))\{C_i x(t) + m(t)\} \tag{32}$$

where $G_i = A_i - LC_i$, and $\overline{y}(t)$ is the transmitted signal. The overall transmitter consists of message embedded chaotic system (31) and fuzzy signal masking system (32). The modulation form (31) and (32) can be regarded as an extension of modulated chaotic communications. To recover the message,

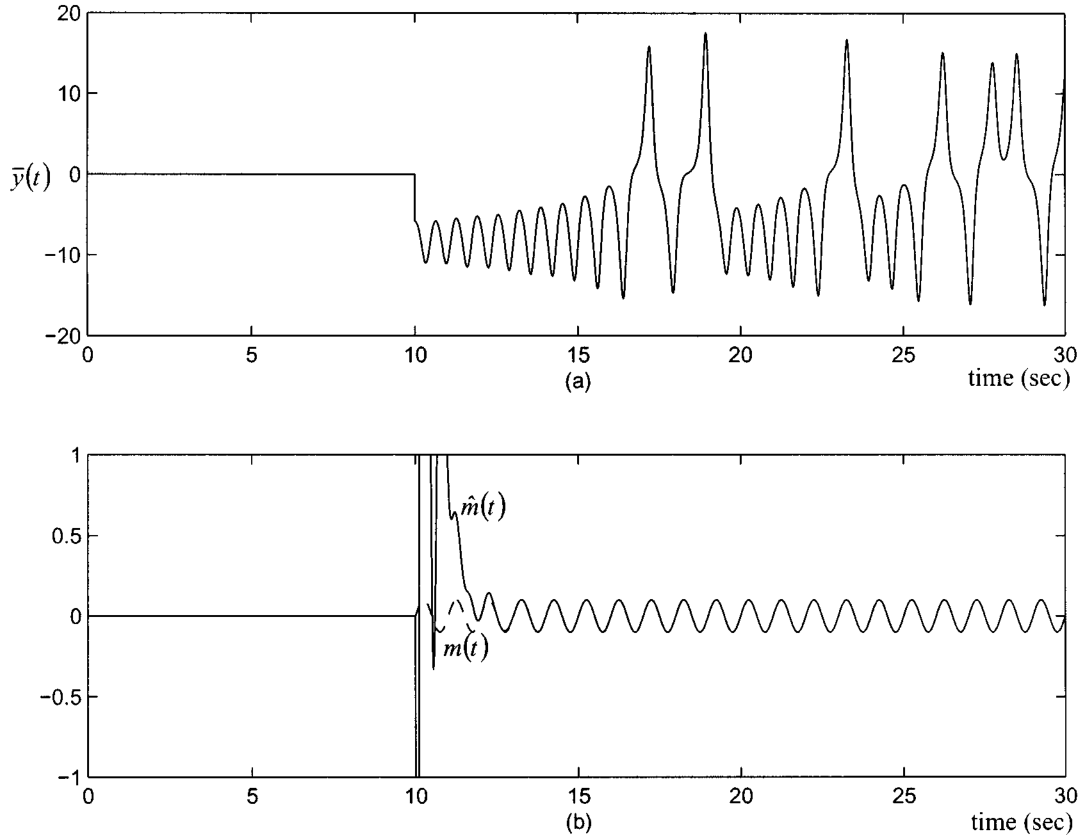


Fig. 11. (a) Coupling signal $\overline{y}(t)$; (b) original message $m(t)$ and recovered message $\hat{m}(t)$ using Lorenz's system with crisp signal masking activated at $t \geq 10$.

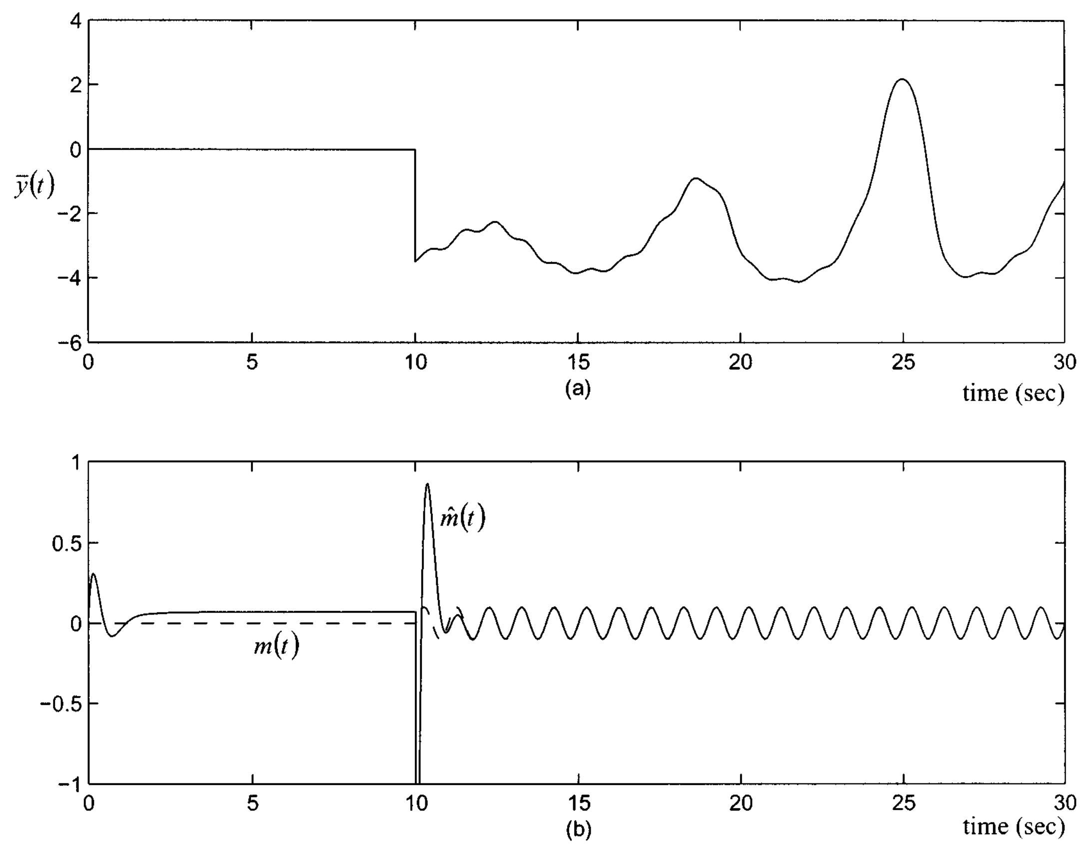


Fig. 12. (a) Coupling signal $\overline{y}(t)$; (b) original message $m(t)$ and recovered message $\hat{m}(t)$ using transformed Rössler's system with crisp signal masking activated at $t \geq 10$.

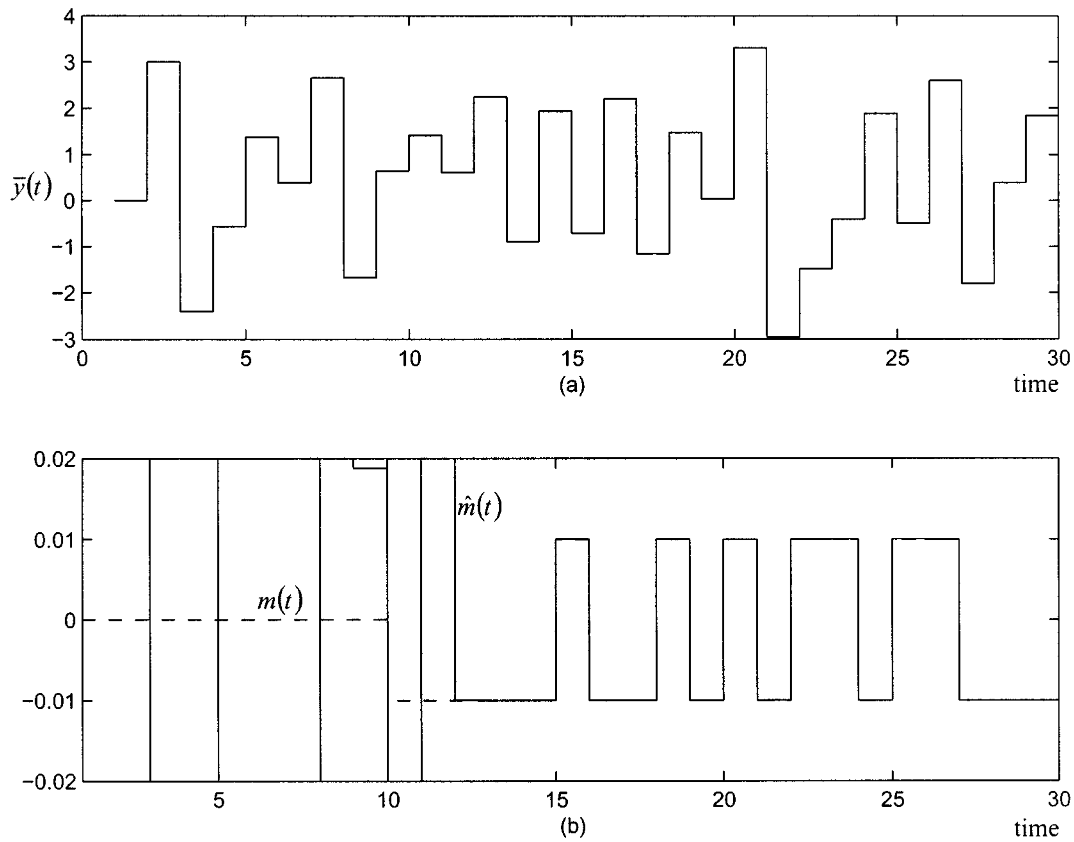


Fig. 13. (a) Coupling signal $\overline{y}(t)$; (b) original message $m(t)$ and recovered message $\hat{m}(t)$ using Lozi map with crisp signal masking activated at $t \geq 10$.

the fuzzy receiver is designed as (11) with $\overline{y}(t)$ instead of $y(t)$, which yields the error dynamics:

$$s\tilde{x}(t) = \sum_{i=1}^{r} \mu_i(z(t))G_i x(t) - \sum_{i=1}^{r} \mu_i(\hat{z}(t))G_i\hat{x}(t) \tag{33}$$

$$\tilde{y}(t) = \sum_{i=1}^{r} \mu_i(z(t))C_i x(t) - \sum_{i=1}^{r} \mu_i(\hat{z}(t))C_i\hat{x}(t) + m(t) \tag{34}$$

where $\tilde{x}(t) \equiv x(t) - \hat{x}(t)$, and $\tilde{y}(t) \equiv \overline{y}(t) - \hat{y}(t)$.

The conditions for ensuring the message is recovered are given in the following theorem.

Theorem 6: If the EL conditions (14) in Theorem 1 are satisfied, then the overall error dynamics (33) becomes

$$s\tilde{x}(t) = G\tilde{x}(t),$$

where $G = G_1 = G_i$, for $i = 1, 2, \cdots, r$. The error system described by the above equation for CFS or DFS is stabilized if the corresponding Theorems 2 and 3 are satisfied. Therefore $\tilde{y}(t) \to m(t)$ as $t \to \infty$. □

The proofs for CFS and DFS are the same as Theorems 2 and 3, respectively. It is noted that when $\tilde{x}(t)$ converges to zero as $t \to \infty$, then $\lim_{t\to\infty}(z - \hat{z}) = 0$ in (34), and $\lim_{t\to\infty} \tilde{y}(t) = m(t)$.

To enhance the convergence rate of recovering the message, the decay rate of errors is carefully considered. The following LMI's design problems for CFS and DFS are performed according to Theorem 6.

Chaotic Communication with Fuzzy Signal Masking for Decay Rate—CFS: See the first equation at the bottom of the next page, where $M_i = C_i P^{-1}$ and $X = P^{-1}$. This yields that (17) becomes $\dot{V}(\tilde{x}(t)) \leq -2\alpha V(\tilde{x}(t))$ with parameter α tuning the decay rate.

Chaotic Communication with Fuzzy Signal Masking for Decay Rate—DFS: The second equation at the bottom of the next page shows where $M_i = C_i P^{-1}$ and $X = P^{-1}$. Equation (20) becomes $\Delta V(\tilde{x}(t)) \leq -(1-\beta)V(\tilde{x}(t))$ with parameter β tuning the decay rate.

B. Crisp Signal Masking

To extract the information from the transmitted signal in fuzzy signal masking, the EL conditions must be kept for the design. Without restricting common bias terms, another method utilizing crisp (typical) signal masking is proposed to be simpler. By introducing the synchronization scheme with crisp driving signal, the chaotic transmitter which has crisp signal masking mechanism can be represented as a T–S fuzzy model as

$Transmitter\ Rule\ i$:

$$\begin{aligned}&\text{IF } \overline{y}(t) \text{ is } F_i \text{ THEN}\\ &sx(t) = A_i x(t) + \eta_i(t) + L_i m(t)\\ &\overline{y}(t) = Cx(t) + m(t), \qquad i = 1, 2, \cdots, r\end{aligned}$$

where the gains L_i, $i = 1, 2, \cdots, r$, will be determined later. The fuzzy inferred result for chaotic transmitter is obtained, that is

$$\begin{aligned} sx(t) &= \sum_{i=1}^{r} \mu_i(\overline{y}(t)) \left\{ \overline{A}_i x(t) + \eta_i(t) + L_i \overline{y}(t) \right\} \\ \overline{y}(t) &= Cx(t) + m(t) \end{aligned}$$

where $\overline{A}_i = A_i - L_i C$. Let us design the receiver as (23) with $\overline{y}(t)$ instead of $y(t)$, which yields the error system:

$$\begin{aligned} s\tilde{x}(t) &= \sum_{i=1}^{r} \mu_i(\overline{y}(t)) \overline{A}_i \tilde{x}(t) \\ \tilde{y}(t) &= C\tilde{x}(t) + m(t). \end{aligned} \tag{35}$$

Theorem 7: The error system represented by the fuzzy inferred system (35) for CFS or DFS is uniformly asymptotically stable if there exist the gains L_i such that the corresponding Theorems 4 and 5 are satisfied, respectively. Meanwhile, $\tilde{y}(t)$ converges to $m(t)$ as $t \to \infty$. □

The conditions for ensuring information recovered are derived using Lyapunov method. Similar as the above section, the decay rate design for CFS and DFS communications are performed by solving LMI's problems as follows:

Chaotic Communication with Crisp Signal Masking for Decay Rate—CFS:

$$\begin{aligned} &\underset{N_i,\, X}{maximize}\, \alpha \\ &\text{subject to } P > 0,\ \alpha > 0 \\ &\quad -A_i^T P - PA_i + C^T N_i^T + N_i C - 2\alpha P > 0, \ \text{ for all } i \end{aligned}$$

where $N_i \equiv PL_i$. This yields that (28) becomes $\dot{V}(\tilde{x}(t)) \leq -2\alpha V(\tilde{x}(t))$ with parameter α tuning the decay rate.

Chaotic Communication with Crisp Signal Masking for Decay Rate—DFS:

$$\begin{aligned} &\underset{N_i,\, X}{\text{minimize}}\, \beta \\ &\text{subject to } P > 0,\ 0 < \beta < 1 \\ &\quad \begin{bmatrix} \beta P & (PA_i - N_i C)^T \\ PA_i - N_i C & P \end{bmatrix} > 0, \ \text{ for all } i \end{aligned}$$

where $N_i \equiv PL_i$. The equation (30) becomes $\Delta V(\tilde{x}(t)) \leq -(1-\beta)V(\tilde{x}(t))$ with parameter β tuning the decay rate.

Using the above proposed methods to solve gains, a trade-off exists, that is, the fuzzy signal masking technique may induce large coupling signal amplitude due to large values of C_i. On the other hand, the crisp signal masking technique may destroy the original chaotic signal due to large values of L_i. However, these phenomenons may not always occur even when optimal decay rate is pursued. For the examples in the following Section VI, instead of using the optimal gains (C_i or L_i), the parameters α and β are tuned to obtain a mild gain C_i or L_i whereas suitable magnitude for coupling signals or chaotic characteristics can be sustained.

VI. Numerical Simulations of Typical Chaotic Systems

To show the validity of proposed synchronization and secure communications, we give in the following numerical examples on both discrete-time and continuous-time chaotic systems.

Example 1: Using the fuzzy driving signal, the synchronization for Chua's circuit, Rössler's system, and Henon map is considered. The initial values of $x(0)$ are set different form those of $\hat{x}(0)$ and the fuzzy response system is activated at $t \geq 10$ (second). Figs. 2–4 show the synchronization results for the cor-

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$$\begin{aligned} &\underset{M_i,\, X}{\text{minimize}}\, \varepsilon \\ &\underset{M_i,\, X}{\text{maximize}}\, \alpha \\ &\text{subject to } X > 0,\ \varepsilon > 0,\ \alpha > 0 \\ &\quad -A_i X - XA_i^T + M_i^T L^T + LM_i - 2\alpha X > 0, \ \text{ for all } i, \\ &\begin{bmatrix} \varepsilon I & \{A_1 X - LM_1 - (A_i X - LM_i)\}^T \\ A_1 X - LM_1 - (A_i X - LM_i) & I \end{bmatrix} > 0, \ \text{ for } 2 \leq i \leq r \end{aligned}$$

$$\begin{aligned} &\underset{M_i,\, X}{\text{minimize}}\, \varepsilon \\ &\underset{M_i,\, X}{\text{minimize}}\, \beta \\ &\text{subject to } X > 0,\ \varepsilon > 0,\ 0 < \beta < 1 \\ &\quad \begin{bmatrix} \beta X & (A_i X - LM_i)^T \\ A_i X - LM_i & X \end{bmatrix} > 0, \ \text{ for all } i, \\ &\begin{bmatrix} \varepsilon I & \{A_1 X - LM_1 - (A_i X - LM_i)\}^T \\ A_1 X - LM_1 - (A_i X - LM_i) & I \end{bmatrix} > 0, \ \text{ for } 2 \leq i \leq r \end{aligned}$$

responding chaos, respectively. It can be seen that $\tilde{x}(t) \to 0$ as $t \to \infty$.

Example 2: Using the crisp driving signal, the synchronization for the chaotic systems, such as Lorenz's equation, transformed Rössler's system, and Lozi map, is considered. The simulation conditions are set up as same as Example 1. Figs. 5–7 show the synchronization results for the corresponding system, respectively. It can be seen that $\tilde{x}(t) \to 0$ as $t \to \infty$.

Example 3: Using the fuzzy signal masking, the secure communications employing Chua's circuit, Rössler's system, and Henon map, are considered. The message $m(t)$ is a sine wave and is considered to be low powered. The initial values of $x(0)$ are set different from those of $\hat{x}(0)$ and the fuzzy response system is activated at $t \geq 10$ (second). The coupling signal $\overline{y}(t)$, and the original message $m(t)$ and recovered message $\hat{m}(t)$ of corresponding systems are shown in Figs. 8–10.

Example 4: Using the crisp signal masking, the secure communications to Lorenz's system, transformed Rössler's system, and Lozi map, are considered. The simulation conditions are set up as same as Example 3. The coupling signal $\overline{y}(t)$, and the original message $m(t)$ and recovered message $\hat{m}(t)$ of corresponding systems are shown in Figs. 11–13.

VII. Conclusions

In this work, a synthesis of fuzzy model-based designs for chaotic synchronization and communication has been proposed. The T–S fuzzy models for continuous and discrete chaotic systems were exactly derived with only one premise variable. Following the general fuzzy models, the fuzzy driving signal and crisp driving signal are employed with a natural and simple way due to two properties, namely, common bias terms and the same premise variable and driving signal. Then the asymptotic synchronization is achieved by solving EL or non-EL LMI's design problems. As an application of fuzzy model-based synchronization, the secure communications of chaotic systems using fuzzy/crisp signal masking are proposed in the same design framework to recover messages asymptotically. The advantage of this synthesis design is that all well-known chaotic systems stated in Section I can achieve their applications on synchronization and secure communications by using either the fuzzy driving signal or the crisp driving signal. Numerical simulations are shown to be consistent with theoretical statements.

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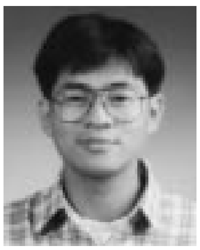

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