IV. CONCLUSION

The various models associated with the Chua's circuit have been recently found helpful in demonstrating and explaining the various facets of chaos [3]. Attempts are made in the literature to discover new members of the large family of Chua's circuits and to find their close relatives [8], [9]. An example for such a recently discovered relative is described in [10], where a nonautonomous related circuit is discussed. (The latter circuit can also be regarded as a relative of the RL diode circuit). The present communication deals with the well known Colpitts oscillator, which is shown to be topologically similar to Chua's circuit. It is also shown here that when the nonlinearity of the active device in the Colpitts oscillator is modified to be purely odd, then the circuit exhibits chaotic phenomena closely related to those exhibited by the classical [3] Chua's circuit. Hence, yet another relative of Chua's circuit has been discovered.

A reviewer has pointed out that recent works of Chua et al. [8], [9] have established mathematically the exactly detailed relationship between Chua's oscillator and relatively many other 3-D systems. The works prove that such 3-D systems are topologically conjugate to Chua's oscillator (or in circuit terms, they are equivalent to Chua's oscillator [11]). A Chua's oscillator is obtained by adding a resistor in series with the inductor in Chua's circuit [11]. The classical Chua's oscillator [11] is, therefore, conjugate to the circuit in Fig. 1(b). Hence, by demonstrating that there exists a robust relationship between the Colpitts oscillator in Fig. 1(a) and the Chua's oscillator [11], one can show that the two systems of Fig. 1 are not simply loose relatives, but they are even strongly related and can be regarded as being conjugate one to the other. Reference [9], which is strongly related to [8], cites an example due to Arneodo et al. [12] of a 3-D system that is strictly proved in [9] as being conjugate to the Chua's oscillator. It is interesting that the latter example ([10] in [9]) is in fact the same equation as the one that represents the presently discussed Colpitts oscillator (3). Hence, we can conclude that the two member systems of Fig. 1 are not merely related, but they are even conjugate one to the other. Therefore, due to the helpful remarks of the reviewer, we are assured that the answer to the question posed in the title of the present communication is strongly yes.

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On the Relationship Between the Chaotic Colpitts Oscillator and Chua's Oscillator

Michael Peter Kennedy

Abstract—In this letter, we show that the two-region third-order piecewise-linear dynamics of the chaotic Colpitts oscillator may be mapped to a Chua's oscillator with an asymmetric nonlinearity.

I. INTRODUCTION

It has recently been shown that the dynamics of a chaotic Colpitts oscillator (shown in Fig. 1(a)) can be captured by a third-order autonomous circuit model containing just one nonlinear element—a two-segment piecewise-linear resistor (Fig. 1(b)), [1].

The circuit is described by a system of three autonomous state equations

\[ \begin{align*}
C_1 \frac{dV_{CE}}{dt} &= I_L - I_C \\
C_2 \frac{dV_{BE}}{dt} &= -\frac{V_{EE} + V_{BE}}{R_{EE}} - I_L - I_B \\
L \frac{dI_C}{dt} &= V_{CC} - V_{CE} + V_{BE} - I_L R_L.
\end{align*} \]

We model the transistor as a two-segment piecewise-linear voltage-controlled resistor \( R_N \) and a linear current-controlled current source. Thus

\[ I_B = \begin{cases} 0 & \text{if } V_{BE} \leq V_{TH} \\ V_{TH} - V_{BE} & \text{if } V_{BE} > V_{TH} \\ \beta I_B & \text{if } V_{BE} > V_{TH} \end{cases} \]

where \( V_{TH} \) is the threshold voltage (\( \approx 0.75 \) V), \( R_N \) is the small-signal on-resistance of the base-emitter junction, and \( \beta \) is the forward current gain of the device.

Fig. 2 shows a chaotic attractor in this two-region piecewise-linear oscillator.

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In this letter, we address the question: what is the connection between this two-region third-order piecewise-linear dynamical system and Chua’s oscillator [2] (shown in Fig. 3)? We will show that the chaotic Colpitts oscillator may be mapped to a Chua’s oscillator with the same eigenvalue pattern if the Chua diode [3] is allowed to have an asymmetric driving-point characteristic.

II. CHUA’S OSCILLATOR

Chua et al. have shown that almost every continuous third-order odd-symmetric three-region piecewise-linear vector field \( F' \) may be mapped onto a Chua’s oscillator (whose vector field \( F \) is topologically conjugate to \( F' \)) by means of the following algorithm [2]:

1. Calculate the eigenvalues \( (\mu_1, \mu_2, \mu_3) \) and \( (\nu_1, \nu_2, \nu_3) \) associated with the linear and affine regions, respectively, of the vector field \( F' \) of the circuit or system whose attractor is to be reproduced (up to linear conjugacy) by Chua’s oscillator.
2. Find a set of circuit parameters \( \{C_1, C_2, L, R, R_0, G_a, G_b, E\} \) so that the resulting eigenvalues \( \mu_j \) and \( \nu_j \), for Chua’s oscillator satisfy \( \mu_j = \mu'_j \) and \( \nu_j = \nu'_j \), \( j = 1, 2, 3 \).

III. PIECEWISE-LINEAR STRUCTURE OF THE CHAOTIC COLPITTOS OSCILLATOR

Cutoff region \( (V_{BE} \leq V_{TH}) \):

In the cutoff region, the chaotic Colpitts oscillator shown in Fig. 1(b) is described by

\[
\begin{align*}
\frac{dV_{CE}}{dt} &= \frac{1}{C_1} I_L \\
\frac{dV_{BE}}{dt} &= -\frac{1}{R_{EE} C_2} V_{BE} - \frac{1}{C_2} I_L - \frac{V_{EE}}{R_{EE} C_2} \\
\frac{dI_L}{dt} &= \frac{1}{L} V_{CE} + \frac{1}{L} V_{BE} - \frac{R_L}{L} I_L + \frac{V_{CC}}{L}.
\end{align*}
\]

The equilibrium point in this region is given by

\[
\begin{bmatrix}
V_{CE} \\
V_{BE} \\
I_L
\end{bmatrix} = \begin{bmatrix}
V_{CC} - V_{EE} \\
-V_{EE} \\
0
\end{bmatrix}.
\]

If \( -V_{EE} > V_{TH} \), then this equilibrium point lies outside the cutoff region and is called a virtual equilibrium point of the system [4]; in fact, it lies in the forward active region.

Forward active region \( (V_{BE} > V_{TH}) \):

In the BJTs forward active region of operation, the circuit is described by

\[
\begin{align*}
\frac{dV_{CE}}{dt} &= -\frac{\beta_f}{R_{ON} C_1} V_{BE} + \frac{1}{C_1} I_L + \frac{\beta_f V_{TH}}{R_{ON} C_1} \\
\frac{dV_{BE}}{dt} &= \left(-\frac{1}{R_{EE} C_2} + \frac{1}{R_{ON} C_2}\right) V_{BE} - \frac{1}{C_2} I_L \\
&\quad - \frac{V_{EE}}{R_{EE} C_2} + \frac{V_{TH}}{R_{ON} C_2} \\
\frac{dI_L}{dt} &= -\frac{1}{L} V_{CE} + \frac{1}{L} V_{BE} - \frac{R_L}{L} I_L + \frac{V_{CC}}{L}.
\end{align*}
\]

This region has an equilibrium point defined by

\[
\begin{bmatrix}
V_{CE} \\
V_{BE} \\
I_L
\end{bmatrix} = \begin{bmatrix}
V_{CC} + \frac{-V_{EE} R_{ON} (1 + \beta_f) R_E V_{TH} + \beta_f R_L (V_{EE} + V_{TH})}{R' - \frac{-V_{EE} R_{ON} (1 + \beta_f) R_E V_{TH}}{R' + \beta_f R_L (V_{EE} + V_{TH})}} \\
-V_{EE} \\
0
\end{bmatrix},
\]

where \( R' = R_{ON} + (1 + \beta_f) R_{EE} \). This equilibrium point lies in the forward active region.

Thus, the chaotic Colpitts oscillator [1] has a true equilibrium point in its forward active region and a virtual equilibrium point associated with the cutoff region, also lying in the forward active region.

Chua’s oscillator always possesses an equilibrium point at the origin (in the so-called \( D_0 \) region) because the driving-point characteristic of its nonlinear resistor goes through the origin. In addition, it has a symmetrically-placed pair of true or virtual equilibrium points associated with the outer regions [4].
Therefore, we identify the forward active region with the \( D_0 \) region of Chua’s oscillator and the cutoff region with the outer regions (\( D_{-1} \) and \( D_1 \)) and attempt to map the dynamics of the chaotic Colpitts oscillator to Chua’s oscillator by following the algorithm presented in [2].

IV. FROM EIGENVALUES TO CIRCUIT PARAMETERS

**Step 1:**

Let \( \{ p_1, p_2, p_3, q_1, q_2, q_3 \} \) be the equivalent eigenvalue parameters defined by

\[
\begin{align*}
p_1 &= \mu' + \mu_2 + \mu_3 \\
p_2 &= \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 \\
p_3 &= \mu_1\mu_2\mu_3 \\
q_1 &= \nu'_1 + \nu'_2 + \nu'_3 \\
q_2 &= \nu'_1\nu'_2 + \nu'_2\nu'_3 + \nu'_3\nu'_1 \\
q_3 &= \nu'_1\nu'_2\nu'_3
\end{align*}
\]

where \((\mu'_1, \mu'_2, \mu'_3)\) and \((\nu'_1, \nu'_2, \nu'_3)\) denote the eigenvalues associated with the vector fields in the \( D_0 \) and \( D_1 \) regions, respectively.

The eigenvalues of the forward active region are the roots of the characteristic polynomial

\[
\lambda^3 + \left( \frac{G_a}{C_2} + \frac{R_1}{L} \right) \lambda^2 + \left( \frac{G_a R_L}{C_2} + \frac{1}{L C_1} + \frac{1}{L C_2} \right) \lambda + \frac{G_a' \lambda}{L C_1 C_2} = 0
\]

where \( G_a = 1/R_{EE} + 1/R_{ON} \) and \( G_a' = 1/R_{EE} + (1 + \beta_f)/R_{ON} \).

We label these roots \( \mu'_1, \mu'_2, \) and \( \mu'_3 \). The eigenvalues of the cutoff region in the Colpitts oscillator are defined as the roots of the characteristic polynomial

\[
\lambda^3 + \left( \frac{G_0}{C_2} + \frac{R_1}{L} \right) \lambda^2 + \left( \frac{G_0 R_L}{C_2} + \frac{1}{L C_1} + \frac{1}{L C_2} \right) \lambda + \frac{G_0' \lambda}{L C_1 C_2} = 0
\]

where \( G_0 = G_0' = 1/R_{EE} \). We label these eigenvalues \( \nu'_1, \nu'_2, \) and \( \nu'_3 \).

**Step 2:**

Define

\[
\begin{align*}
k_1 &= -a_0 + \left( \frac{b_0 - b_1}{a_1} \right) \left( p_1 - \frac{b_2 - b_3}{a_1} \right) \\
k_2 &= p_2 - \left( \frac{b_0 - b_1}{a_1} \right) \left( p_1 - \frac{b_2 - b_3}{a_1} \right) \\
k_3 &= -\left( \frac{b_0 - b_1}{a_1} \right) - \frac{\alpha}{k_2} \\
k_4 &= -k_1 k_3 + k_2 \left( \frac{b_0 - b_1}{a_1} \right)
\end{align*}
\]

**Step 3:**

The corresponding normalized circuit parameters are given by

\[
\begin{align*}
C_1 &= 1 \\
\frac{C_2}{C_1} &= -\frac{k_2}{k_4} \\
\frac{L}{L} &= -k_2 \\
\frac{R}{R_0} &= \frac{1 - \frac{k_2}{k_4}}{k_2} \\
\frac{G_a}{G_0} &= -\frac{p_1 + \left( \frac{b_0 - b_1}{a_1} \right)}{\frac{b_0 - b_1}{a_1}} + \frac{\alpha}{k_2} \\
\frac{G_b}{G_0} &= -\frac{q_1 + \left( \frac{b_0 - b_1}{a_1} \right)}{\frac{b_0 - b_1}{a_1}} + \frac{\alpha}{k_2}
\end{align*}
\]

This equation is a corrected version of (23) in [2].

**Step 4:**

The breakpoints \( \pm E \) of the piecewise-linear Chua diode can be chosen arbitrarily since the choice of \( E \) does not affect either the eigenvalues or the dynamics; it simply scales the circuit voltages.

![Fig. 4. Periodic attractor in Chua’s oscillator](image)

V. SIMULATION RESULTS

Throughout the following discussion, we consider a single set of model parameters \( V_{CC} = 5 \) V, \( R_1 = 35 \) \( \Omega \), \( L = 98.5 \) \( \mu \)H, \( C_1 = 54 \) nF, \( C_2 = 54 \) nF, \( R_{EE} = 400 \) \( \Omega \), and \( V_{EE} = 5 \) V. The BJT is specified by three parameters: \( V_{TH} = 0.75 \) V, \( R_{ON} = 100 \Omega \), and \( \beta_f = 200 \).

The eigenvalues in the forward active and cutoff regions are (to ten significant figures)

\[
\begin{align*}
\mu'_1 &= -2042294.350 \\
\mu'_2 &= 727741.4596 + j1703284.985 \\
\mu'_3 &= 727741.4596 - j1703284.985 \\
\nu'_1 &= -22674.2934 \\
\nu'_2 &= -189475.9761 + j589887.5958 \\
\nu'_3 &= -189475.9761 - j589887.5958
\end{align*}
\]

With \( k_f = 0.01 \) and \( k_E = 10^3 \), the corresponding parameters for Chua’s oscillator (to ten significant figures) are

\[
\begin{align*}
C_1 &= 1.00000000000 \text{pF} \\
C_2 &= 39.23427816 \text{nF} \\
L &= 6.747352901 \text{mH} \\
R &= 8.253831974 \text{k} \Omega \\
R_o &= 3.139471850 \text{m} \Omega \\
G_a &= -120.9243561 \mu \text{S} \\
G_b &= -121.1095302 \mu \text{S}
\end{align*}
\]

Fig. 4 shows the attractor of Chua’s oscillator for this parameter set with \( E = 1 \) V. This periodic attractor, which lies in all three regions of the state space, is qualitatively different from the original chaotic Colpitts attractor, which was confined to just two regions.
The problem may be removed by breaking the symmetry of the nonlinearity to ensure that the trajectory is confined to the inner and just one of outer regions. In particular, we can move the left breakpoint to the right so that the steady-state trajectory never enters the $D_1$ region. The modified Chua diode nonlinearity, defined explicitly by

$$I_R = f(V_R)$$

$$= \frac{1}{2}(G_0 - G_b)\left[E^+ - E^-\right] + G_b V_R$$

$$+ \frac{1}{2}(G_0 - G_b)\left[|V_R + E^-| - |V_R - E^+|\right]$$

is shown in Fig. 5. A piecewise-linear voltage-controlled nonlinear resistor of this type may be implemented using the synthesis procedure described in [5]. We refer to Chua’s oscillator with an asymmetric nonlinearity as an asymmetric Chua’s oscillator.

With $E^- = 0.02$ V and $E^+ = 1$ V, the resulting attractor in the asymmetric Chua’s oscillator is confined to the $D_2$ and $D_3$ regions and is chaotic (see Fig. 6).

VI. CONCLUDING REMARKS

The dynamics of the continuous two-region piecewise-linear Colpitts oscillator may be mapped onto an equivalent asymmetric Chua’s oscillator by choosing the breakpoints of the asymmetric Chua diode such that the resulting attractor is confined to two regions of the state space. The dynamics of the circuits are equivalent in the sense that the vector field of the Colpitts oscillator is linearly conjugate to that of the asymmetric Chua’s oscillator.

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