On Bounds of the Parametric Range of Bifurcation of Chua’s Circuit

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Abstract—This brief presents general representations for bounds on the parametric range of the $G$, $C_1$, $L$, and $C_2$-bifurcations of Chua’s circuit. The critical points for the appearance of bifurcations are derived. Consequently, chaos can be quenched by adjusting parameters to meet these conditions.

I. INTRODUCTION

Chaotic regions for Chua’s circuit were considered in [1]. The authors of reference [2] derived the correct transfer function and obtained the maximum possible parametric range of $G$- and $C_1$-bifurcations. Using this information one can take full advantage of the range to accomplish many kinds of tasks. However, the results above are based on specific parameter calculation, and no exact formula is given. Furthermore, one wishes to know the conditions for which chaos disappears to avoid it. According to the two-parameter bifurcation diagram in the $\alpha$-$\beta$ plane, given in [3], one sees that chaos can be quenched by making $\alpha$ sufficiently small or adjusting $\beta$ appropriately when $\alpha$ is fixed. However, no quantitative formula is given. In this paper, in addition to $G$- and $C_1$-bifurcations, the $L$- and $C_2$-bifurcations are also discussed. At the same time we obtained the general expressions of the maximum bifurcation range of the $G$, $C_1$, $L$, and $C_2$-bifurcations, and derived the criteria for the disappearance of chaos in Chua’s circuit by making the maximum bifurcation range zero. By using the above method, similar criteria for the other members of Chua’s circuit family can also be obtained. So both goals: to use the circuit for chaos generation and to avoid it, are accomplished.

II. STATE EQUATION AND TRANSFER FUNCTION

Chua’s circuit is shown in Fig. 1(a) and the piecewise-linear characteristic of the nonlinear Chua’s diode is shown in Fig. 1(b). The state equation of the circuit can be written in the form

$$\frac{dx(t)}{dt} = Ax(t) + Bg(C^T x(t))$$  \hspace{1cm} (1)

where

$$A = \begin{bmatrix} G & G & 0 \\ \frac{1}{C_2} & \frac{1}{C_2} & 0 \\ 0 & - \frac{1}{L} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

III. RESULTS

Fig. 2. The absolute stability sector in the $G$-$m$ plane.

and

$$x(t) = [v_{C1}(t) \ v_{C2}(t) \ i_L(t)]^T$$  \hspace{1cm} (3)

The transfer function of the linear part can be written as [2]

$$G(s) = \frac{V(s)}{I(s)} = \frac{1}{C_1 C_2 L} \left( \frac{s^2 + \frac{G}{C_2} s + \frac{1}{C_2 L}}{s^2 + \frac{1}{C_1 C_2} G + \frac{1}{C_1 C_2 L}} \right)$$  \hspace{1cm} (4)

From (1) the characteristic equation of the linearized system is given by

$$s^3 + \left( \frac{G}{C_2} + \frac{G + m_1}{C_1} \right) s^2 + \frac{1}{C_1 C_2 L} \left( \frac{G + m_1}{C_1 C_2} + \frac{G}{C_2} \right) s + \frac{G + m_1}{C_1 C_2 L} = 0$$  \hspace{1cm} (5)

where $G = 1/R$, $m_1 = m_1$ (inside region), and $m_1 = m_0$ (outside region). Let $a = 1/C_1$, $b = 1/C_2$, and $l = 1/L$. Then, (5) can be rewritten as

$$s^3 + (aG + bG + am_1) s^2 + (bl + abGm_1) s + abl(G + m_1) = 0$$  \hspace{1cm} (6)
Applying the Routh–Hurwitz criterion to (6), we get the following conditions for the absolute stability of the linearized system.

Condition A: \( aG + bG + am_i > 0 \)

Condition B: \( bl + abGM_i > 0 \)

Condition C: \( abl(G + m_i) > 0 \)

Condition D: \( a^2m_i^2 + (a + b)GM_i + bl > 0 \).

In this paper, assume \( a, b, l \) are positive real numbers, and \( G \geq 0, m_i \leq 0 \). Now we will consider the \( G_1, C_1, L_1, \) and \( C_2 \)-bifurcations individually.

### III. The G-Bifurcation

Here, the bifurcation parameter is \( G \). So, according to (7)–(10), there exist four boundaries \( C_1, C_2, C_3, \) and \( C_4 \), as shown in Fig. 2. In Fig. 2, shaded areas denote regions where the linearized system is stable. According to Ogorzalek’s conjecture, the second stability sector (below curve BA in Fig. 2) is necessary for chaos existence. It is easy to show that the boundaries given by (8)–(10) intersect at the point \( A = \left( \sqrt{I/a}, -\sqrt{I/a} \right) \). And, from the equality of (10),

\[
G = -a^2m_i^2 - bl/a(a + b)m_i.
\]

Let \( dG/dm_i = 0 \). Then, we obtain the coordinates of one of the extreme value points, \( B = \left( (2\sqrt{bI})/(a + b), -(\sqrt{bI}/a) \right) \).

All these coordinates of the points \( A \) and \( B \) are shown in Fig. 2. Obviously, the maximum possible dynamic range of \( G \)-bifurcation is from point \( A \) to point \( B \) in horizontal direction, for example, when \( m_i = -\sqrt{I/a} \) and \( m_0 = -\sqrt{bI/a} \). In [2], the authors first found the possible maximum dynamic range by using a specific parameter value. Here we will use mathematical formula to obtain the exact representation.

Let the three eigenvalues of the linearized Chua’s circuit operating at boundary \( C_4 \) be \( \gamma, \sigma \pm j\omega \). It can be proved that \( \sigma = 0 \) on the boundary \( C_4 \) above point \( A \). We can call this boundary a Hopf bifurcation curve. Then the extreme value points \( A \) and \( B \) exhibit the maximum dynamic range of \( G \)-bifurcation when point \( B \) is the intersection of \( C_4 \) and \( m_0 \). In the following derivation, we use the stability and the eigenvalues of the linearized system.

Substituting (11), i.e., the relationship between \( G \) and \( m_i \) into (6), we have

\[
s^3 + (aG + bG + am_i)s^2 + (bl + abGM_i)s + abl(G + m_i)
= s^3 - \frac{bl}{am_i}s^2 + \frac{ab(l - am_i^2)}{a + b}s - \frac{b^2l(l - am_i^2)}{(a + b)m_i}
= \left( s - \frac{bl}{am_i} \right) \left( s^2 + \frac{ab(l - am_i^2)}{a + b} \right) = 0.
\]

From (13) indeed we have a pure imaginary eigenvalue, which completes our derivation.

For the sake of comparison, let us choose the same values as in [2], i.e., \( a = 1/C_1, b = 1/C_2, l = 1/L_1, \) and \( G \) is variable. For extreme value point \( B, m_0 = -0.29397237 \). The corresponding eigenvalues are as follows:

\[
\begin{align*}
\gamma &= -\sqrt{bI} = -\sqrt{I/a} = -2.645713 \\
\sigma &= 0 \\
\omega &= \pm \sqrt{bI/(a + b)} = \pm \sqrt{2.3664319} \\
\end{align*}
\]

The corresponding eigenvalues at different \( m_1 \) when \( \sigma = 0 \) are given in Table I.

For point \( A, m = -\sqrt{I/a} = -\sqrt{I/9} = -0.8819171 \), we have

\[
\begin{align*}
\gamma &= -b\sqrt{I/a} = -0.8819171 \\
\sigma &= 0 \\
\omega &= 0 \\
\end{align*}
\]

And therefore, point \( A \) is a critical bifurcation point.

In the above, we discussed the maximum bifurcation range. Next we will discuss when this bifurcation range is zero. By adjusting parameters to move boundary \( C_4 \) toward the right direction, one can finally make point \( B \) coincide with point \( A, \) i.e., the horizontal distance between point \( A \) and \( B \) becomes equal to zero. In this case, we have

\[
\sqrt{I/a} = \frac{2\sqrt{bI}}{a + b}.
\]
and
\[ \sqrt{\frac{I}{\alpha}} = -\frac{\sqrt{bl}}{\alpha}. \] (17)

Finally, we obtain the relationship \( a = b. \)

According to custom, \( \alpha = \frac{C_2}{C_1}. \) Therefore, \( \alpha = a/b = 1. \)

And from \( \beta = \frac{C_4}{(LC)^2}, \) and \( G = \sqrt[4]{l} \) at point \( A, \) so we have \( \beta = a/b = 1. \) It is not difficult to show that when \( \alpha < 1 \) or \( \beta < 1 \) no bifurcation appears at all. Obviously, \( \alpha = 1 \) or \( \beta = 1 \) are the criteria for when chaos disappears. Then, chaos can be quenched by adjusting the parameters to either make \( \alpha < 1, \) or \( \beta < 1. \) In fact, this criterion is meaningful both in theory and practice [4].

IV. THE C1-BIFURCATION

The stability conditions is the same as (7)–(10). Here, the bifurcation parameter is \( a = 1/C_1. \)

The corresponding boundaries are C5–C8, as shown in Fig. 3. It is easy to show that (8)–(10) intersect at the point \( C = (l/G^2, -G). \)

In a similar way, for equality of (10), let \( \delta a/\delta m_1 = 0 \) and we have the coordinates of point \( D = (2\sqrt{bl}/G - b, -\sqrt{bl}G/(2\sqrt{bl} - bG)). \)

So the maximum dynamic range of the C1-bifurcation is from point \( C \) to point \( D \) in the horizontal direction.

In boundary C8, the eigenvalues are given by
\[ \begin{align*}
\gamma &= -(aG + bG + am_1) \\
\sigma &= 0 \\
\omega &= \pm \sqrt{bl + abGm_1}
\end{align*} \] (18)

From (18) indeed we have a pure imaginary eigenvalue, which completes our derivation.

In the same way, we choose the same values as in [2], i.e.,
\( b = 1/C_3 = 1, l = 1/L = 1, G = 0.7, \) and \( a = 1/C_1 \) is variable.

For extreme value point \( D, m_0 = \sqrt{blG}/(2\sqrt{bl} - bG) = -0.40355944 \) and \( a = 2\sqrt{bl}/G - b = 6.5922895. \) The corresponding eigenvalues are as follows:
\[ \begin{align*}
\gamma &= -\sqrt{bl} - \sqrt{\tau} = -2.6457513 \\
\sigma &= 0 \\
\omega &= \pm \sqrt{(l - \sqrt{blG})} \pm 2.6289147
\end{align*} \] (19)

For corresponding eigenvalues when \( \sigma = 0 \) at different \( m_0, \) see Table II.

For point \( C, m_0 = -G = -0.7 \) and \( a = 1/G^2 = 14.285714, \) we have
\[ \begin{align*}
\gamma &= -0.7 \\
\sigma &= 0 \\
\omega &= 0
\end{align*} \] (20)

Therefore, point \( C \) is the critical bifurcation point in the case when \( C_1 \) is the bifurcation parameter.

By adjusting parameters to move curve C8 toward the right direction, one can finally make point \( D \) coincide with point \( C, \) i.e., the horizontal distance between point \( C \) and \( D \) becomes equal to zero. This is the minimum range we need. Here we have
\[ \frac{l}{G^2} = \frac{2\sqrt{bl} - b}{G} \] (21)

and
\[ \frac{-\sqrt{bl}}{2\sqrt{bl} - bG} = -G. \] (22)

Finally, we have \( G = \sqrt{bl} \) or \( l = bG^2. \) Therefore \( \beta < 1 \) is the criterion for chaos disappearance.

V. THE L-BIFURCATION

The stability conditions is the same as (7)–(10). Now we select the bifurcation parameter \( l = 1/L. \)

The corresponding boundaries C9–C12 are shown in Fig. 4. It is easy to show that (8)–(10) intersect at the point \( E = (aG^2, -G), \) as shown in Fig. 4.

In a similar way, for equality of (10), we have
\[ l = -\frac{a^2 m_1^2 + a(a + b)Gm_1}{b}. \] (23)

Let \( dl/\delta m_1 = 0 \) and we have the coordinates of point \( F = \{(a + b)^2 G^2/4b, -(a + b)G/2b\}, \) as shown in Fig. 4.

So the maximum dynamic range of the L-bifurcation is from point \( E \) to point \( F \) in the horizontal direction.

In boundary C12, the eigenvalues are given by
\[ \begin{align*}
\gamma &= -(aG + bG + am_1) \\
\sigma &= 0 \\
\omega &= \pm \sqrt{-a^2 m_1(m_1 + G)}
\end{align*} \] (24)
TABLE III

<table>
<thead>
<tr>
<th>Bifurcation Parameters</th>
<th>Equilibrium $P^\pm$ $m_0 = -0.388,888,889$</th>
<th>Equilibrium $P^\pm$ $m_0 = -0.5$</th>
<th>Equilibrium $P^0$ $m_1 = -0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 1/L$</td>
<td>$\gamma$</td>
<td>$\sigma \pm j\omega$</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>$12.25$</td>
<td>$-3.5$</td>
<td>$0 \pm j,3.130,495$</td>
<td>$1.282,300 - 0.541,150 \pm j,2.881,838$</td>
</tr>
<tr>
<td>$11.25$</td>
<td>$-2.5$</td>
<td>$0 \pm j,2.846,05$</td>
<td>$1.317,829 - 0.558,913 \pm j,2.714,907$</td>
</tr>
</tbody>
</table>

From (24) indeed we have a pure imaginary eigenvalue, which completes our derivation.

In the same way, we choose the same values as in [2], i.e., $a = 1/C_4 = 9$, $b = 1/C_2 = 1$, $G = 0.7$, and $l = 1/L$ is variable, we have for the extreme value point $E$, $m_0 = -(a + b)G/(2a) = -0.388\,888\,889$. The corresponding eigenvalues are as follows:

$$
\begin{align*}
\gamma &= \frac{(a + b)G}{2} = -3.5 \\
\sigma &= 0 \\
\omega &= \pm \frac{G}{2} \sqrt{a^2 - b^2} = \pm 3.130\,495\,2
\end{align*}
$$

(25)

For corresponding eigenvalues when $\sigma = 0$ at different $m_0$, see Table III.

For point $E$, $m_0 = -0.7$, we have

$$
\begin{align*}
\gamma &= -0.7 \\
\sigma &= 0 \\
\omega &= 0
\end{align*}
$$

(26)

Therefore, point $E$ is the critical bifurcation point in the case when $L$ is the bifurcation parameter.

By adjusting parameters to move boundary $C_{12}$ toward the right direction, one can finally make point $F$ coincide with point $E$, i.e., the horizontal distance between point $E$ and $F$ becomes equal to zero. This is the minimum range we need. Here we have

$$
(a + b)^{\,2}G^2 = \frac{aG^2}{4b}
$$

(27)

and

$$
-(a + b)G = -G.
$$

(28)

Finally, we have $a = b$. And we have $l = aG^2$ at point $E$. Similarly, $\alpha < 1$ or $\beta < 1$ is the criterion for chaos disappearance. This criteria are the same as those obtained in $G$- and $C_1$-bifurcation.

VI. The $C_2$-Bifurcation

The stability condition is the same as (7)-(10). Now we select the bifurcation parameter $b = 1/C_2$.

The corresponding boundaries $C_{13}$-$C_{16}$ are shown in Fig. 5. It is easy to show that (7)-(10) intersect at the point $M = (0, -G)$, as shown in Fig. 5. In a similar way, for equality of (10), we have

$$
b = -\frac{a^2m_i(m_i + G)}{aGm_i + l}.
$$

(29)

Let $db/dm_i = 0$ and we have the coordinates of point $N = \{2l - \sqrt{l(l - aG^2)} - aG^2\}/G^2, -l - \sqrt{l(l - aG^2)}/(aG)\}$, as shown in Fig. 5.

So the maximum dynamic range of the $C_2$-bifurcation is from point $M$ to point $N$ in the horizontal direction.

In boundary $C_{16}$, the eigenvalues are given by

$$
\begin{align*}
\gamma &= -(aG + bG + am_i) \\
\sigma &= 0 \\
\omega &= \pm \sqrt{b[l + aGm_i]}
\end{align*}
$$

(30)

This completes our derivation.

In the same way, we choose the same element values as in [2], i.e., $a = 1/C_4 = 9$, $l = 1/L = 7$, $G = 0.7$, and $b = 1/C_2$ is variable.

For extreme value point $N$, $m_0 = \{2l - \sqrt{l(l - aG^2)}\}/(aG) = -0.435\,248\,62$ and $b = \{2l - 2\sqrt{l(l - aG^2)} - aG^2\}/G^2 = 2.192\,107\,1$

The corresponding eigenvalues are as follows:

$$
\begin{align*}
\gamma &= \frac{-2l - 2\sqrt{l(l - aG^2)} - l - \sqrt{l(l - aG^2)}}{G} \\
\sigma &= 0 \\
\omega &= \pm \frac{\sqrt{2l - 2\sqrt{l(l - aG^2)} - aG^2}}{G^2} \sqrt{l(l - aG^2)}
\end{align*}
$$

(31)

For the corresponding eigenvalues when $\sigma = 0$ at different $m_0$, see Table IV.

For point $M$, $m_0 = -G = -0.7$ and $b = 0$, we have

$$
\begin{align*}
\gamma &= 0 \\
\sigma &= 0 \\
\omega &= 0
\end{align*}
$$

(32)

Therefore, in this case point $M$ is a critical bifurcation point.

By adjusting parameters to move curve $C_{16}$ toward right, point $N$ will coincide with point $M$ eventually, i.e., the horizontal distance between point $N$ and $M$ is equal to zero. Then we have

$$
\frac{l - \sqrt{l(l - aG^2)}}{aG} = -G
$$

(33)

and

$$
\frac{2l - \sqrt{l(l - aG^2)} - aG^2}{G^2} = 0.
$$

(34)

From (33) we have $l = aG^2$. Therefore $\beta = a/b$ is the criterion for chaos disappearance. Unfortunately, from (34) we have $a = 0$, or $G = 0$. However, from taking the limit of $b$, we can obtain the result. Let point $m_0$ between $M$ and $N$ of the boundary $C_{16}$.

Fig. 5. The absolute stability sector in the $b$-$m$ plane.
the coordinates of $m$ are $\{-(kG, ka^2G^2(1-k)/(1-kaG^2))\}$, where $k < 1$. Now we have

$$\alpha = \frac{C_2}{C_3}.$$  

So $\alpha < 1$ or $\beta < 1$ is the criterion for chaos disappearance.

**VII. Discussion**

The bounds of the dynamic range of bifurcation of Chua's circuit are derived. From the exact representation of the bounds, we obtain not only maximum dynamic range of bifurcation, but also obtain the condition of chaos disappearance when dynamic range of bifurcation is 0. In fact, there is a simpler way to determine the conditions to stop bifurcation, and that is to set $\omega = 0$ at points B, D, F, and N. The result is the same as the above derivation. The results mentioned above are instructive both in theory and in application.

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**References**


