On Bounds of the Parametric Range of Bifurcation of Chua's Circuit

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***Abstract*—This brief presents general representations for bounds on the parametric range of the G-, C_1-, L-, and C_2-bifurcations of Chua's circuit. The critical points for the appearance of bifurcations are derived. Consequently, chaos can be quenched by adjusting parameters to meet these conditions.**

I. Introduction

Chaotic regions for Chua's circuit were considered in [1]. The authors of reference [2] derived the correct transfer function and obtained the maximum possible parametric range of G- and C_1-bifurcations. Using this information one can take full advantage of the range to accomplish many kinds of tasks. However, the results above are based on specific parameter calculation, and no exact formula is given. Furthermore, one wishes to know the conditions for when chaos disappears to avoid it. According to the two-parameter bifurcation diagram in the α–β plane, given in [3], one sees that chaos can be *quenched* by making α sufficiently small or adjusting β appropriately when α is fixed. However, no quantitative formula is given. In this paper, in addition to G- and C_1-bifurcations, the L- and C_2-bifurcations are also discussed. At the same time we obtained the general expressions of the maximum bifurcation range of the G-, C_1-, L-, and C_2-bifurcations, and derived the criteria for the disappearance of chaos in Chua's circuit by making the maximum bifurcation range zero. By using the above method, similar criteria for the other members of Chua's circuit family can also be obtained. So both goals: to use the circuit for chaos generation and to avoid it, are accomplished.

II. State Equation and Transfer Function

Chua's circuit is shown in Fig. 1(a) and the piecewise-linear characteristic of the nonlinear Chua's diode is shown in Fig. 1(b). The state equation of the circuit can be written in the form

$$\frac{dx(t)}{dt} = Ax(t) + Bg[C^T x(t)] \tag{1}$$

where

$$A = \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{C_1} \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

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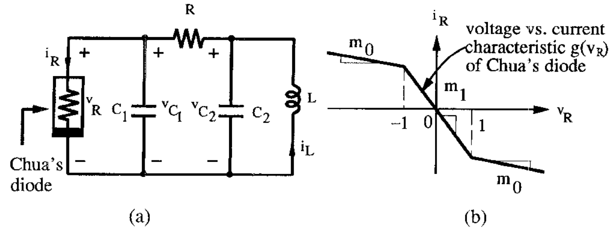

Fig. 1. Chua's circuit.

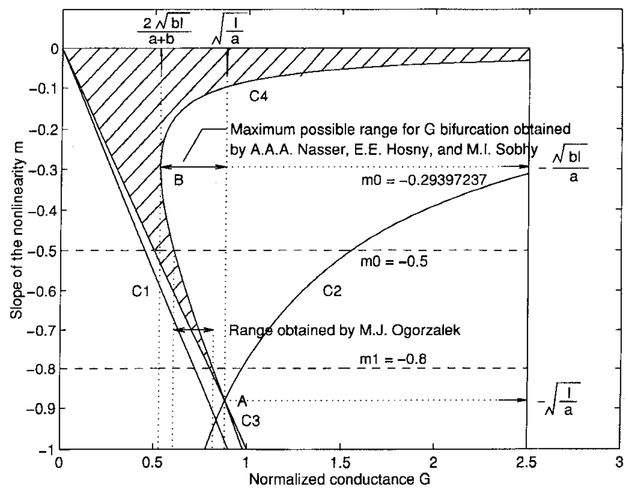

Fig. 2. The absolute stability sector in the G–m plane.

and

$$x(t) = [v_{C1}(t) \quad v_{C2}(t) \quad i_L(t)]^T \tag{3}$$

The transfer function of the linear part can be written as [2]

$$\left.\begin{aligned} G(s) &= \frac{V(s)}{I(s)} \\ &= C^T(sI - A)^{-1}B \\ &= \frac{\frac{1}{C_1}\left(s^2 + \frac{G}{C_2}s + \frac{1}{C_2 L}\right)}{s^3 + \frac{(C_1 + C_2)G}{C_1 C_2}s^2 + \frac{1}{C_2 L}s + \frac{G}{C_1 C_2 L}} \end{aligned}\right\}. \tag{4}$$

From (1) the characteristic equation of the linearized system is given by

$$s^3 + \left(\frac{G}{C_2} + \frac{G + m_i}{C_1}\right)s^2 + \left(\frac{1}{C_2 L} + \frac{G m_i}{C_1 C_2}\right)s + \frac{G + m_i}{C_1 C_2 L} = 0 \tag{5}$$

where $G = 1/R$, $m_i = m_1$ (inside region), and $m_i = m_0$ (outside region).

Let $a = 1/C_1$, $b = 1/C_2$, and $l = 1/L$. Then, (5) can be rewritten as

$$s^3 + (aG + bG + am_i)s^2 + (bl + abGm_i)s + abl(G + m_i) = 0. \tag{6}$$

TABLE I
EIGENVALUES OF THE CHARACTERISTIC EQUATION AT EQUILIBRIUM P^0 AND $P^{\pm}$

Bifurcation parameters	**Equilibrium $P^{\pm}$ $m_0 = -0.293\,972\,37$**		**Equilibrium $P^{\pm}$ $m_0 = -0.5$**		**Equilibrium P^0 for both choices $m_1 = -0.8$**	
G	γ	$\sigma \pm j\omega$	γ	$\sigma \pm j\omega$	γ	$\sigma \pm j\omega$
0.529 150 26	−2.645 751	$0 \pm j2.366\,432$				
0.605 555 56			−1.555 556	$0 \pm j2.067\,607$		
0.817 222 22					−0.972 221	$0 \pm j1.056\,409$

Applying the Routh–Hurwitz criterion to (6), we get the following conditions for the absolute stability of the linearized system.

$$\textbf{ConditionA}: \quad aG + bG + am_i > 0 \tag{7}$$

$$\textbf{ConditionB}: \quad bl + abGm_i > 0 \tag{8}$$

$$\textbf{ConditionC}: \quad abl(G + m_i) > 0 \tag{9}$$

$$\textbf{ConditionD}: \quad a^2m_i^2 + a(a+b)Gm_i + bl > 0. \tag{10}$$

In this paper, assume a, b, and l are positive real numbers, and $G \geq 0$, $m_i \leq 0$. Now we will consider the G-, C_1-, L-, and C_2-bifurcations individually.

III. THE G-BIFURCATION

Here, the bifurcation parameter is G. So, according to (7)–(10), there exist four boundaries **C1**, **C2**, **C3**, and **C4**, as shown in Fig. 2. In Fig. 2, shaded areas denote regions where the linearized system is stable. According to Ogorzalek's conjecture, the second stability sector(below curve BA in Fig. 2) is necessary for chaos existence. It is easy to show that the boundaries given by (8)–(10) intersect at the point $A = (\sqrt{l/a}, -\sqrt{l/a})$. And, from the equality of (10),

$$G = \frac{-a^2{m_i}^2 - bl}{a(a+b)m_i}. \tag{11}$$

Let $dG/dm_i = 0$. Then, we obtain the coordinates of one of the extreme value points, $B = [(2\sqrt{bl})/(a+b), (-\sqrt{bl})/a]$.

All these coordinates of the points A and B are shown in Fig. 2. Obviously, the maximum possible dynamic range of G-bifurcation is from point A to point B in horizontal direction, for example, when $m_1 = -\sqrt{l/a}$ and $m_0 = -\sqrt{bl}/a$. In [2], the authors first found the possible maximum dynamic range by using a specific parameter value. Here we will use mathematical formula to obtain the exact representations.

Let the three eigenvalues of the linearized Chua's circuit operating at boundary **C4** be γ, $\sigma \pm j\omega$. It can be proved that $\sigma = 0$ on the boundary **C4** above point A. We can call this boundary a Hopf bifurcation curve. Then the extreme value points A and B exhibit the maximum dynamic range of G-bifurcation when point B is the intersection of **C4** and m_0. In the following derivation, we use the stability and the eigenvalues of the linearized system.

Substituting (11), i.e., the relationship between G and m_i into (6), we have

$$\left.\begin{aligned} & s^3 + (aG + bG + am_i)s^2 + (bl + abGm_i)s \\ & \quad + abl(G + m_i) \\ = \; & s^3 - \frac{bl}{am_i}s^2 + \frac{ab(l - a{m_i}^2)}{a+b}s - \frac{b^2l(l - am_i^2)}{(a+b)m_i} \\ = \; & \left(s - \frac{bl}{am_i}\right)\left[s^2 + \frac{ab(l - am_i^2)}{a+b}\right] \\ = \; & 0 \end{aligned}\right\}. \tag{12}$$

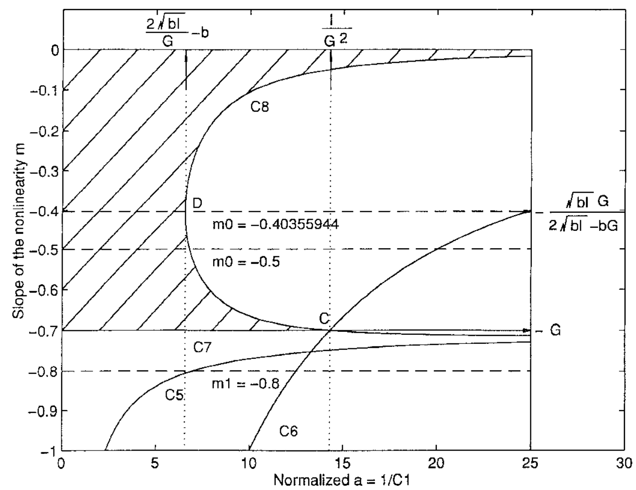

Fig. 3. The absolute stability sector in the a–m plane.

Now we have

$$\left.\begin{aligned} \gamma &= \frac{bl}{am_i} \\ \sigma &= 0 \\ \omega &= \pm\sqrt{\frac{ab(l - am_i^2)}{a+b}} \end{aligned}\right\}. \tag{13}$$

From (13) indeed we have a pure imaginary eigenvalue, which completes our derivation.

For the sake of comparison, let us choose the same values as in [2], i.e., $a = 1/C_1 = 9$, $b = 1/C_2 = 1$, $l = 1/L = 7$, and G is variable. For extreme value point B, $m_0 = -0.293\,972\,37$. The corresponding eigenvalues are as follows:

$$\left.\begin{aligned} \gamma &= -\sqrt{bl} = -\sqrt{7} = -2.6457513 \\ \sigma &= 0 \\ \omega &= \pm\sqrt{bl\,\frac{a-b}{a+b}} = \pm\sqrt{5.6} = \pm 2.366\,431\,9 \end{aligned}\right\}. \tag{14}$$

The corresponding eigenvalues at different m_i when $\sigma = 0$ are given in Table I.

For point A, $m = -\sqrt{l/a} = -\sqrt{7/9} = -0.881\,917\,1$, we have

$$\left.\begin{aligned} \gamma &= -b\sqrt{\frac{l}{a}} = -0.8819171 \\ \sigma &= 0 \\ \omega &= 0 \end{aligned}\right\}. \tag{15}$$

And therefore, point A is a critical bifurcation point.

In the above, we discussed the maximum bifurcation range. Next we will discuss when this bifurcation range is zero. By adjusting parameters to move boundary **C4** toward the right direction, one can finally make point B coincide with point A, i.e., the horizontal distance between point A and B becomes equal to zero. In this case, we have

$$\sqrt{\frac{l}{a}} = \frac{2\sqrt{bl}}{a+b} \tag{16}$$

TABLE II
EIGENVALUES OF THE CHARACTERISTIC EQUATION AT EQUILIBRIUM P^0 AND $P^{\pm}$

Bifurcation parameters	**Equilibrium** $P^{\pm}$ $m_0 = -0.403\,559\,44$		**Equilibrium** $P^{\pm}$ $m_0 = -0.5$		**Equilibrium** P^0 **for both choices** $m_1 = -0.8$	
$a = 1/C_1$	γ	$\sigma \pm j\omega$	γ	$\sigma \pm j\omega$	γ	$\sigma \pm j\omega$
6.559 289 5	−2.644 390	$0 \pm j\ 2.268\,712$			1.033 844	$-0.538\,958 \pm j\ 2.037\,331$
6.797 660 5			−2.059 532	$0 \pm j\ 2.149\,609$	1.083 893	$-0.552\,064 \pm j\ 2.021\,210$

and

$$-\sqrt{\frac{l}{a}} = -\frac{\sqrt{bl}}{a}. \tag{17}$$

Finally, we obtain the relationship $a = b$.

According to custom, $\alpha \triangleq C_2/C_1$. Therefore, $\alpha = a/b = 1$. And from $\beta \triangleq C_2/(LG^2)$, and $G = \sqrt{l/a}$ at point A, so we have $\beta = a/b = 1$. It is not difficult to show that when $\alpha < 1$ or $\beta < 1$ no bifurcation appears at all. Obviously, $\alpha = 1$ or $\beta = 1$ are the criteria for when chaos disappears. Then, chaos can be quenched by adjusting the parameters to either make $\alpha < 1$, or $\beta < 1$. In fact, this criterion is meaningful both in theory and practice [4].

IV. THE C_1-BIFURCATION

The stability conditions is the same as (7)–(10). Here, the bifurcation parameter is $a = 1/C_1$.

The corresponding boundaries are **C5–C8**, as shown in Fig. 3. It is easy to show that (8)–(10) intersect at the point $C = (l/G^2, -G)$. In a similar way, for equality of (10), let $da/dm_i = 0$ and we have the coordinates of point $D = \{2\sqrt{bl}/G - b, -\sqrt{bl}G/(2\sqrt{bl} - bG)\}$.

So the maximum dynamic range of the C_1-bifurcation is from point C to point D in the horizontal direction.

In boundary **C8**, the eigenvalues are given by

$$\left.\begin{aligned} \gamma &= -(aG + bG + am_i) \\ \sigma &= 0 \\ \omega &= \pm\sqrt{bl + abGm_i} \end{aligned}\right\}. \tag{18}$$

From (18) indeed we have a pure imaginary eigenvalue, which completes our derivation.

In the same way, we choose the same values as in [2], i.e., $b = 1/C_2 = 1$, $l = 1/L = 7$, $G = 0.7$, and $a = 1/C_1$ is variable.

For extreme value point D, $m_0 = -\sqrt{bl}G/(2\sqrt{bl} - bG) = -0.403\,559\,44$ and $a = 2\sqrt{bl}/G - b = 6.559\,289\,5$. The corresponding eigenvalues are as follows:

$$\left.\begin{aligned} \gamma &= -\sqrt{bl} = -\sqrt{7} = -2.6457513 \\ \sigma &= 0 \\ \omega &= \pm\sqrt{b(l - \sqrt{bl}G)} = \pm 2.268\,914\,7 \end{aligned}\right\}. \tag{19}$$

For corresponding eigenvalues when $\sigma = 0$ at different m_0, see Table II.

For point C, $m_0 = -G = -0.7$ and $a = l/G^2 = 14.285\ 714$, we have

$$\left.\begin{aligned} \gamma &= -0.7 \\ \sigma &= 0 \\ \omega &= 0 \end{aligned}\right\}. \tag{20}$$

Therefore, point C is the critical bifurcation point in the case when C_1 is the bifurcation parameter.

By adjusting parameters to move curve **C8** toward the right direction, one can finally make point D coincide with point C, i.e., the horizontal distance between point C and D becomes equal to zero. This is the minimum range we need. Here we have

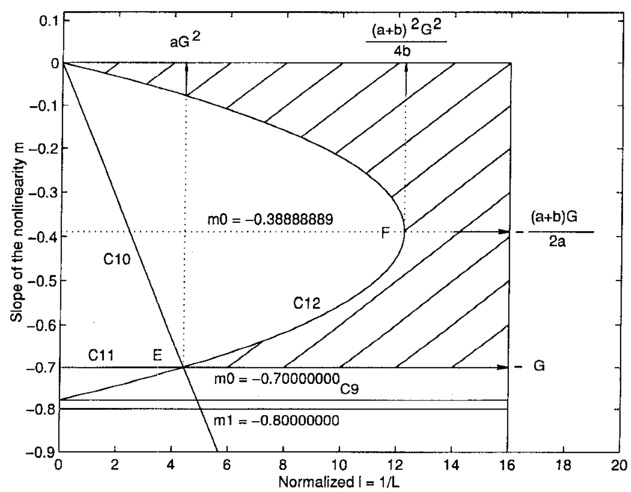

Fig. 4. The absolute stability sector in the l–m plane.

$$\frac{l}{G^2} = \frac{2\sqrt{bl}}{G} - b \tag{21}$$

and

$$-\frac{\sqrt{bl}G}{2\sqrt{bl} - bG} = -G. \tag{22}$$

Finally, we have $G = \sqrt{l/b}$ or $l = bG^2$. Therefore $\beta < 1$ is the criterion for chaos disappearance.

V. The L-Bifurcation

The stability conditions is the same as (7)–(10). Now we select the bifurcation parameter $l = 1/L$.

The corresponding boundaries **C9–C12** are shown in Fig. 4. It is easy to show that (8)–(10) intersect at the point $E = (aG^2, -G)$, as shown in Fig. 4.

In a similar way, for equality of (10), we have

$$l = -\frac{a^2 m_i{}^2 + a(a+b)Gm_i}{b}. \tag{23}$$

Let $dl/dm_i = 0$ and we have the coordinates of point $F = \{(a+b)^2G^2/4b, -(a+b)G/2a\}$, as shown in Fig. 4.

So the maximum dynamic range of the L-bifurcation is from point E to point F in the horizontal direction.

In boundary **C12**, the eigenvalues are given by

$$\left.\begin{aligned} \gamma &= -(aG + bG + am_i) \\ \sigma &= 0 \\ \omega &= \pm\sqrt{-a^2 m_i(m_i + G)} \end{aligned}\right\}. \tag{24}$$

TABLE III
EIGENVALUES OF THE CHARACTERISTIC EQUATION AT EQUILIBRIUM P^0 AND $P^{\pm}$

Bifurcation parameters	Equilibrium $P^{\pm}$ $m_0 = -0.388\,888\,89$		Equilibrium $P^{\pm}$ $m_0 = -0.5$		Equilibrium P^0 $m_1 = -0.8$	
$a = 1/L$	γ	$\sigma \pm j\omega$	γ	$\sigma \pm j\omega$	γ	$\sigma \pm j\omega$
12.25	-3.5	$0 \pm j\ 3.130495$			1.282 300	$-0.541\,150 \pm j\ 2.881\,838$
11.25			-2.5	$0 \pm j\ 2.84605$	1.317 829	$-0.558\,913 \pm j\ 2.714\,907$

From (24) indeed we have a pure imaginary eigenvalue, which completes our derivation.

In the same way, we choose the same values as in [2], i.e., $a = 1/C_1 = 9$, $b = 1/C_2 = 1$, $G = 0.7$, and $l = 1/L$ is variable, we have for the extreme value point F, $m_0 = -(a+b)G/(2a) = -0.388\ 888\ 89$. The corresponding eigenvalues are as follows:

$$\left.\begin{aligned} \gamma &= -\frac{(a+b)G}{2} = -3.5 \\ \sigma &= 0 \\ \omega &= \pm\frac{G}{2}\sqrt{a^2-b^2} = \pm 3.130\,495\,2 \end{aligned}\right\}. \tag{25}$$

For corresponding eigenvalues when $\sigma = 0$ at different m_0, see Table III.

For point E, $m_0 = -0.7$, we have

$$\left.\begin{aligned} \gamma &= -0.7 \\ \sigma &= 0 \\ \omega &= 0 \end{aligned}\right\}. \tag{26}$$

Therefore, point E is the critical bifurcation point in the case when L is the bifurcation parameter.

By adjusting parameters to move boundary **C12** toward the right direction, one can finally make point F coincide with point E, i.e., the horizontal distance between point E and F becomes equal to zero. This is the minimum range we need. Here we have

$$\frac{(a+b)^2G^2}{4b} = aG^2 \tag{27}$$

and

$$-\frac{(a+b)G}{2a} = -G. \tag{28}$$

Finally, we have $a = b$. And we have $l = aG^2$ at point E. Similarly, $\alpha < 1$ or $\beta < 1$ is the criterion for chaos disappearance. This criteria are the same as those obtained in G- and C_1-bifurcation.

VI. The C_2-Bifurcation

The stability condition is the same as (7)–(10). Now we select the bifurcation parameter $b = 1/C_2$.

The corresponding boundaries **C13–C16** are shown in Fig. 5. It is easy to show that (7)–(10) intersect at the point $M = (0, -G)$, as shown in Fig. 5. In a similar way, for equality of (10), we have

$$b = -\frac{a^2m_i(m_i+G)}{aGm_i+l}. \tag{29}$$

Let $db/dm_i = 0$ and we have the coordinates of point $N = \{[2l - 2\sqrt{l(l-aG^2)} - aG^2]/G^2, -[l - \sqrt{l(l-aG^2)}]/(aG)\}$, as shown in Fig. 5.

So the maximum dynamic range of the C_2-bifurcation is from point M to point N in the horizontal direction.

In boundary **C16**, the eigenvalues are given by

$$\left.\begin{aligned} \gamma &= -(aG + bG + am_i) \\ \sigma &= 0 \\ \omega &= \pm\sqrt{b(l+aGm_i)} \end{aligned}\right\}. \tag{30}$$

This completes our derivation.

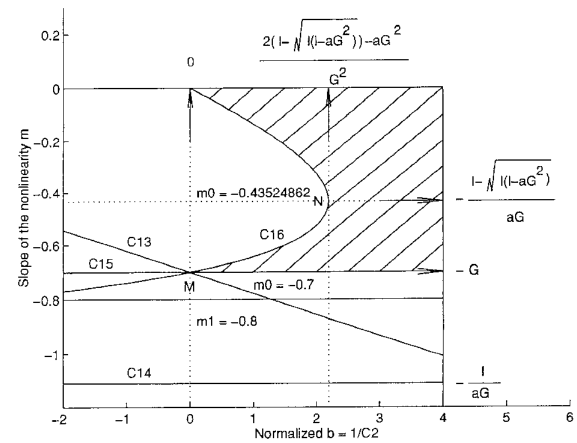

Fig. 5. The absolute stability sector in the b–m plane.

In the same way, we choose the same element values as in [2], i.e., $a = 1/C_1 = 9$, $l = 1/L = 7$, $G = 0.7$, and $b = 1/C_2$ is variable.

For extreme value point N, $m_0 = -[l - \sqrt{l(l-aG^2)}]/(aG) = -0.435\,248\,62$ and $b = [2l - 2\sqrt{l(l-aG^2)} - aG^2]/G^2 = 2.192\ 107\ 1$.

The corresponding eigenvalues are as follows:

$$\left.\begin{aligned} \gamma &= -\frac{2l - 2\sqrt{l(l-aG^2)}}{G} + \frac{l - \sqrt{l(l-aG^2)}}{G} \\ &= -3.917\,237\,1 \\ \sigma &= 0 \\ \omega &= \pm\sqrt{\frac{2l - 2\sqrt{l(l-aG^2)} - aG^2}{G^2}\sqrt{l(l-aG^2)}} \\ &= \pm 3.055\,134\,5 \end{aligned}\right\}. \tag{31}$$

For the corresponding eigenvalues when $\sigma = 0$ at different m_0, see Table IV.

For point M, $m_0 = -G = -0.7$ and $b = 0$, we have

$$\left.\begin{aligned} \gamma &= 0 \\ \sigma &= 0 \\ \omega &= 0 \end{aligned}\right\}. \tag{32}$$

Therefore, in this case point M is a critical bifurcation point.

By adjusting parameters to move curve **C16** toward right, point N will coincide with point M eventually, i.e., the horizontal distance between point N and M is equal to zero. Then we have

$$-\frac{l - \sqrt{l(l-aG^2)}}{aG} = -G \tag{33}$$

and

$$\frac{2l - 2\sqrt{l(l-aG^2)} - aG^2}{G^2} = 0. \tag{34}$$

From (33) we have $l = aG^2$. Therefore $\beta = a/b$ is the criterion for chaos disappearance. Unfortunately, from (34) we have $a = 0$, or $G = 0$. However, from taking the limit of b, we can obtain the result. Let point m between M and N of the boundary **C16**. Then

TABLE IV
EIGNEVALUES OF THE CHARACTERISTIC EQUATION AT EQUILIBRIUM P^0 AND $P^{\pm}$

Bifurcation parameters	**Equilibrium** $P^{\pm}$ $m_0 = -0.435\,248\,62$		**Equilibrium** $P^{\pm}$ $m_0 = -0.5$		**Equilibrium** P^0 **for both choices** $m_1 = -0.8$	
$b = 1/C_2$	γ	$\sigma \pm j\omega$	γ	$\sigma \pm j\omega$	γ	$\sigma \pm j\omega$
2.192 107 1	$-3.917\,237$	$0 \pm j\ 3.055\,136$			1.684 004	$-1.159\,239 \pm j\ 2.618\,591$
2.103 896 1			$-3.272\,727$	$0 \pm j\ 2.846\,050$	1.677 844	$-1.125\,286 \pm j\ 2.575\,554$

the coordinates of m are $\{-kG, ka^2G^2(1-k)\ /(l-kaG^2)\}$, where $k < 1$. Now we have

$$\alpha = \frac{C_2}{C_1}.$$

So $\alpha < 1$ or $\beta < 1$ is the criterion for chaos disappearance.

VII. DISCUSSION

The bounds of the dynamic range of bifurcation of Chua's circuit are derived. From the exact representation of the bounds, we obtain not only maximum dynamic range of bifurcation, but also obtain the condition of chaos disappearance when dynamic range of bifurcation is 0. In fact, there is a simpler way to determine the conditions to stop bifurcation, and that is to set $\omega = 0$ at points B, D, F, and N. The result is the same as the above derivation. The results mentioned above are instructive both in theory and in application.

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