DESIGN OF NONLINEAR OBSERVERS FOR HYPERCHAOS SYNCHRONIZATION USING A SCALAR SIGNAL

Giuseppe Grassi
Dipartimento di Matematica
Università di Lecce, 73100 Lecce, Italy
grassi@ingle01.unile.it

Saverio Mascolo
Dipartimento di Elettrotecnica ed Elettronica
Politecnico di Bari, 70125 Bari, Italy
mascolo@poliba.it

ABSTRACT

In this paper hyperchaos synchronization is restated as a nonlinear observer design issue. This approach leads to a *systematic* tool, which guarantees the synchronization of a *wide class* of hyperchaotic systems via a *scalar* signal. The proposed technique has been applied to synchronize two well-known examples of hyperchaotic dynamics: Rössler's system and Matsumoto-Chua-Kobayashi circuit.

1. INTRODUCTION

In the last few years several researchers have focused their attention on the problems related to chaos synchronization and control [1]-[6], [11]. In particular, different methods have been developed in order to synchronize chaotic systems. For instance, the well-known scheme proposed in [1] consists in taking a chaotic system, duplicating some subsystem and driving the duplicate and the original subsystem with signals from the unduplicated part. When all the Lyapunov exponents of the driven subsystem (response system) are less then zero, the response system synchronizes with the drive system, assuming that both systems start in the same basin of attraction. It should be noted that most of the methods developed until now concerns the synchronization of low dimensional systems, with only one positive Lyapunov exponent. Since this feature limits the complexity of the chaotic dynamics, the adoption of higher dimensional chaotic systems has been recently proposed for secure communications [4]. In fact, the presence of more than one positive Lyapunov exponent clearly improves security by generating more complex dynamics. This approach, however, raises the question of whether synchronization can still be achieved by transmitting a scalar signal. Until now, only some attempts have been made to give an answer to this question. In [4] a linear combination of the original state variables is used to synchronize hyperchaos in Rössler's systems. However, this technique cannot be considered a systematic tool for synchronization, since the coefficients of the linear combination are somewhat arbitrary. An interesting result has been illustrated in [5], where a parameter control method is proposed for hyperchaos synchronization. Anyway, the computation of the Lyapunov exponents is still required in order to verify the synchronization.

In this paper a simple and rigorous method for synchronizing hyperchaotic systems via a scalar signal is developed [6]. The proposed technique, based on nonlinear control theory, has several advantages over the existing methods:

- it enables synchronization to be achieved in a systematic way;
- it can be successfully applied to a wide class of hyperchaotic systems;
- it does not require the computation of any Lyapunov exponent;
- it does not require initial conditions belonging to the same basin of attraction.

The paper is organized as follows. In section 2, hyperchaos synchronization is restated as a nonlinear observer design issue. Following this approach, a linear and time-invariant synchronization error system is obtained, for which a necessary and sufficient condition is given in order to asymptotically stabilize the origin [6]. Finally, in section 3 the proposed method is applied to synchronize two well-known examples of hyperchaotic dynamics: Rössler's system [6] and Matsumoto-Chua-Kobayashi circuit [7] (shortly, the MCK circuit).

2. NONLINEAR OBSERVER DESIGN FOR SYNCHRONIZING HYPERCHAOS

Definition 1: Given two chaotic systems, the dynamics of which are described by the following two sets of differential equations:

$$\dot{x} = f(x) \tag{1}$$
$$\dot{y} = f(y) \tag{2}$$

where $x \in \Re^n$, $y \in \Re^n$, and $f : \Re^n \to \Re^n$ is a nonlinear vector field, systems (1) and (2) are said to be synchronized if

$$e(t) = (y(t) - x(t)) \to 0 \text{ as } t \to \infty \tag{3}$$

where e represents the synchronization error [2].

Definition 2: An observer is a dynamic system designed to be driven by the output of another dynamic system (*plant*) and having the property that the state of the observer converges to the state of the plant [8], [9]. More precisely, given dynamic system (1) with output $z = s(x) \in \Re^n$, the dynamic system

$$\dot{y} = f(y) + g(z - s(y)) \tag{4}$$

is said to be a nonlinear observer of system (1) if its state y converges to x as $t \to \infty$, where $g : \Re^n \to \Re^n$ is a suitably chosen nonlinear function. Moreover, system (4) is said to be a *global* observer of system (1) if $y \to x$ as $t \to \infty$ for any initial condition $y(0)$, $x(0)$ [8].

Remark 1: System (4) is a (global) observer of system (1) if the error system

$$\begin{aligned}\dot{e} &= f(y) + g(s(x) - s(y)) - f(x) \\ &= f(x+e) + g(s(x) - s(x+e)) - f(x) = h(e,t)\end{aligned} \tag{5}$$

has a (globally) asymptotically stable equilibrium point for $e=0$.
A block diagram of a nonlinear observer for the state x of system (1) is reported in Fig. 1.

Assumption 1: The dynamic system (1) can be written as:

$$\dot{x} = f(x) = Ax + b\, f(x) + c \tag{6}$$

where $A \in \Re^{n \times n}, b \in \Re^{n \times 1}, c \in \Re^{n \times 1}$ and $f : \Re^n \to \Re$.

Remark 2: Several hyperchaotic systems satisfy Assumption 1. For example, Rössler's system [6], the MCK circuit [7] and the oscillators reported in [10] all belong to the class defined by (6).

Regarding the synchronizing signal, it is worth noting that $s(x)$ is an artificial output of system (1) which can be properly designed to feed the nonlinear observer (4). Since the adoption of a scalar signal is a suitable feature for secure communications applications, it is assumed that $z = s(x) \in \Re$.

Proposition 1: Given system (6), let

$$s(x) = f(x) + kx \tag{7}$$

be the scalar synchronizing signal with $k = [k_1, k_2, \ldots k_n] \in \Re^{1 \times n}$, and let

$$g(s(x) - s(y)) = b(s(x) - s(y)) \tag{8}$$

be the function g in equation (4). Then system (5) becomes linear and time-invariant, and can be expressed as:

$$\dot{e} = Ae - bke = Ae + bu \tag{9}$$

where $u = -ke$ plays the role of a state feedback.

The proof is reported in [6].

Proposition 2: The n-dimensional linear time-invariant, single-input dynamic system $\dot{x} = Ax + bu$ is controllable if the controllability matrix $[b \;\; Ab \;\; A^2b \;\; \ldots \;\; A^{n-1}b]$ is full rank. In this case, all the eigenvalues are controllable, i.e., they can be arbitrarily assigned by the introduction of state feedback [6].

Now, a necessary and sufficient condition for synchronizing hyperchaos can be given [6].

Proposition 3: Given (6)-(8), a necessary and sufficient condition for the existence of a feedback gain vector k such that system (4) becomes a global observer of system (1) is that all uncontrollable eigenvalues of the error system (9), if any, have negative real parts.
The proof is reported in [6].

3. EXAMPLES

3.1 Synchronization of Rössler's system

Rössler's system [6] can be written in the form (6) as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_1 x_3 + \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \tag{11}$$

This system exhibits a hyperchaotic behavior starting from proper initial conditions (see Fig. 2). Proposition 1 gives:

$$s(x) = x_1 x_3 + \sum_{j=1}^{4} k_j x_j$$

$$g(s(x) - s(y)) = [0 \;\; 0 \;\; 1 \;\; 0]^T (s(x) - s(y))$$

whereas equation (4) becomes:

$$\begin{aligned}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} &= \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} y_1 y_3 + \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \\ &+ [0 \;\; 0 \;\; 1 \;\; 0]^T (s(x) - s(y))\end{aligned} \tag{12}$$

with the error system given by:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = \left(\begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} [k_1 \;\; k_2 \;\; k_3 \;\; k_4] \right) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{13}$$

The controllability matrix of (13) is full rank. Taking into account Propositions 2 and 3, there exists a gain vector $\boldsymbol{k}$ such that system (12) becomes a *global* observer of system (11). This means that $\boldsymbol{y} \to \boldsymbol{x}$ *as* $t \to \infty$ for each initial state $\boldsymbol{x}(0)$ and $\boldsymbol{y}(0)$. For instance, all eigenvalues of (13) are placed in -1 for $\boldsymbol{k}$=[-3.3712 -0.9561 4.3000 -5.8126]. Fig. 3 shows the synchronization between the selected variables x_2 and y_2.

3.2 Synchronization of MCK circuit

The first experimental observation of hyperchaotic oscillations from a *real physical system* has been described by Matsumoto, Chua and Kobayashi [7]. The circuit implemented in [7] is autonomous and contains only one nonlinear element, a three-segment piecewise-linear resistor. All other elements are linear and passive, except an active resistor, which has negative resistance. By considering the parameters and the equations reported in [10], the dynamics of the MCK circuit can be rewritten as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 1.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 10 \\ 0 \end{bmatrix} g(x_1, x_3) \tag{14}$$

$$g(x_1,x_3) = \begin{cases} -0.2 + 3(x_1 - x_3 + 1) & \text{if } x_1 - x_3 < -1, \\ -0.2(x_1 - x_3) & \text{if } -1 \le x_1 - x_3 \le 1, \\ -0.2 + 3(x_1 - x_3 - 1) & \text{if } x_1 - x_3 > 1. \end{cases}$$

The projection of the hyperchaotic attractor on the plane (x_1, x_3) is reported in Fig. 4. From Proposition 1, it follows:

$$s(x) = g(x_1, x_3) + \sum_{i=1}^{n} k_i x_i$$

$$g(s(x) - s(y)) = [-1 \quad 0 \quad 10 \quad 0]^T (s(x) - s(y))$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 1.5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 10 \\ 0 \end{bmatrix} g(y_1, y_3) +$$

$$+ [-1 \quad 0 \quad 10 \quad 0]^T (s(x) - s(y)) \tag{15}$$

Since the controllability matrix of the error system is full rank, its eigenvalues can be moved anywhere. By placing them in -1, it results $\boldsymbol{k}$=[-0.3764 0.2384 0.4324 -0.4314] and system (15) becomes a *global* observer of system (14). The synchronization between the selected variables x_2 and y_2 is shown in Fig. 5.

4. CONCLUSION

In this paper a simple and rigorous method for synchronizing a wide class of hyperchaotic systems via a scalar transmitted signal has been developed. The proposed tool does not require either the computation of the Lyapunov exponents or initial conditions belonging to the same basin of attraction. Simulation results on Rössler's system and MCK circuit have shown the usefulness of the suggested method.

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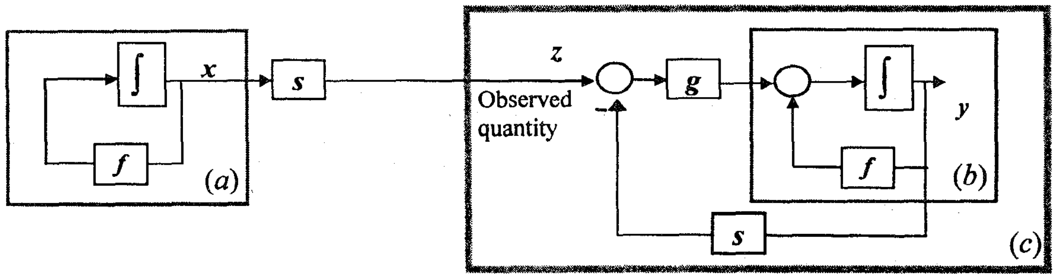

Figure 1. Synchronization as a nonlinear observer issue: (a) system (1); (b) system (2); (c) structure of the observer (4).

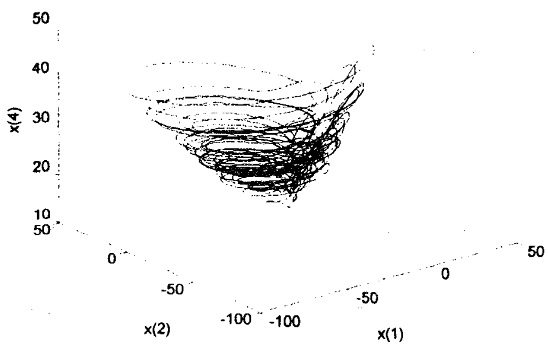

Figure 2. A projection of Rössler's hyperchaotic attractor.

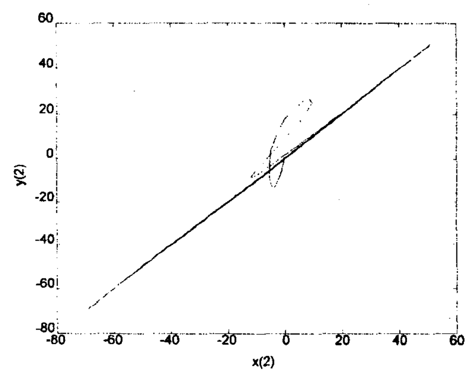

Figure 3. Synchronization between the state variables x_2 and y_2 of systems (11) and (12).

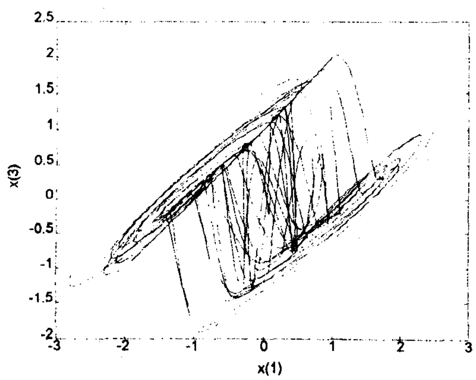

Figure 4. Projection on the plane (x_1, x_3) of the hyperchaotic attractor described by (14).

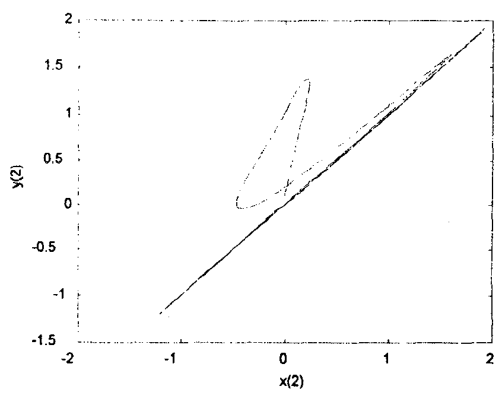

Figure 5. Synchronization between the state variables x_2 and y_2 of systems (14) and (15).