A Spectral Approach for Studying Spatio-Temporal Chaos

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ABSTRACT

A spectral technique is proposed for studying and predicting chaos in a one-dimensional array of Chua's circuits. By use of a Double Fourier Transform the network is reduced to a scalar Lur'e system to which the describing function technique is applied for discovering the existence of periodic wave. Finally by the computation of the distortion index an approximate tool is given for detecting the occurrence of chaos.

I. Introduction

In recent years a great interest has been devoted to the study of the dynamics of networks composed by elementary cells that exhibit a chaotic behaviour (see [1], [3], [4], [6], [7], [8], [9]).

In particular in [3] and [4] a classification of the dynamic phenomena occurring in arrays of chaotic oscillators has been established. Among the electrical engineering community many researchers concentrated on the study of large arrays made of Chua's circuits (that is a simple and robust example of chaotic oscillator, [5]); it has been shown that such arrays may model the propagation failure phenomenon [7] and that may have application in image processing [8], [9].

One of the most interesting behaviours that can be observed in a one-dimensional array of Chua's circuits is the spatio-temporal chaos [6]. Due to the high dimension of such networks only a few analytical tools are available for their study (see [2]).

In this paper we investigate the dynamic behaviour of a chain of Chua's circuits by use of a spectral technique, that represents the extension of the technique introduced by Genesio and Tesi in [10] to systems that have both time and space dependence.

The method is based on the fact that all the cells are identical and therefore by introducing a suitable Double Fourier Transform the network can be reduced to a scalar Lur'e system (see Fig. 2). Then by use of an extension of the describing function technique (see [10]) the existence of periodic waves can be predicted. Finally by computing the distortion index an approximate tool is developed for detecting the occurrence of chaos. The accuracy of the proposed technique has been confirmed by means of time-simulation.

We remark that the advantage of the method is that it permits to predict chaos by means of simple algebraic computations and without performing any simulations (that for large arrays of nonlinear circuits are rather time-consuming); moreover the method is not only applicable to the array of Chua's circuits but to any network composed by identical elementary cells.

II. Chain interconnections of Chua's circuits

We investigate the structure obtained by interconnecting a finite number (M) of classical Chua's circuits [5]: each cell of the chain (shown in Fig.1), composed by two capacitors C_1 and C_2, an inductor L, a conductance G and a Chua's diode ([5]) is coupled to the other cells by means of a conductance G_1.

We will show that the dynamic behaviour of such a structure can be studied by means of a suitable spectral technique; moreover we remark that such a technique can also be applied to the study of similar structures and in particular to the chain interconnection of canonical Chua's circuit introduced in [6] and to the chain of Chua's circuits considered in [7].

The dynamics of the k-th cell of Fig.1 ($1 \leq k \leq M$), after the scaling transformation $\tau = tG/C_2$ and $w_k =$

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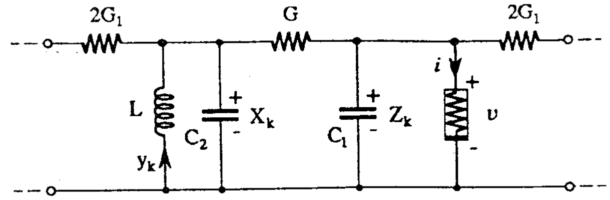

Figure 1: *The fundamental cell of the chain*

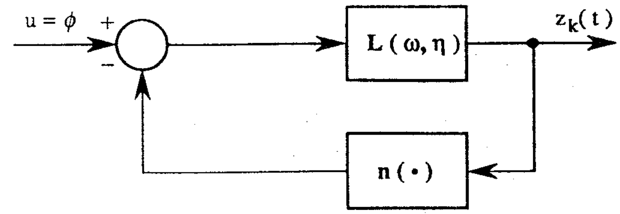

Figure 2: *Chain of Chua's circuits as a Lur'e system*

y_k/G, can be described by:

$$\begin{aligned}\frac{dx_k}{d\tau} &= -(x_k - z_k) + w_k - \gamma(x_k - z_{k-1})\\ \frac{dw_k}{d\tau} &= -\beta x_k\\ \frac{dz_k}{d\tau} &= -\alpha(z_k - x_k) - \alpha n(z_k) - \alpha\gamma(z_k - x_{k+1})\end{aligned} \tag{1}$$

where $\alpha = \frac{C_2}{C_1}$, $\beta = \frac{C_2}{LG^2}$, $\gamma = \frac{G_1}{G}$, $n(z_k) = \frac{1}{G}i(z_k)$.

We assume, for the sake of simplicity, that $z_0 = z_M = 0$, i.e that the first cell is ended by a shortcircuit, in series with the conductance of value $2G_1$ whereas the last cell ($M-1$-th) is ended by the series of the conductance $2G_1$ with the parallel of the inductance L, the capacitor C_2, and the Chua's diode. Moreover, according to [11], we assume that the nonlinearity of the Chua's diode can be approximated by:

$$n(z_k) = -\frac{m}{G}z_k + \frac{k}{G}z_k^3 \tag{2}$$

where the parameters have been fixed to the values $m = 4/5$, $k = 2/45$, $G = 7/10$, which ensure that the $k-th$ Chua's circuit of the chain, if not coupled, has three equilibria located at $z_k = -1.5$, $z_k = 0$, $z_k = 1.5$ (see [11]) In the next section the time variable τ appearing in (1) will be denoted again with t.

III. The spectral method

For studying the above dynamical system we propose a spectral technique based on the introduction of the Double Fourier Transform $F(\omega, \eta)$ (denoted with a capital letter) of functions $f_k(t)$ discrete in space and continuous in time:

$$F(\omega, \eta) = \sum_{k=-\infty}^{k=\infty} \int_{-\infty}^{\infty} f_k(t)\exp(-j\omega\tau)\exp(-j\eta k) \tag{3}$$

By applying the Double Fourier Transform defined above to the set of equations (1) the following relations are obtained:

$$\begin{aligned}j\omega X(\omega,\eta) &= -(1+\gamma)X(\omega,\eta) + W(\omega,\eta)\\ &\quad +(1+\gamma e^{-j\eta})Z(\omega,\eta)\\ j\omega W(\omega,\eta) &= -\beta X(\omega,\eta)\\ j\omega Z(\omega,\eta) &= \alpha(1+\gamma e^{j\eta})X(\omega,\eta) - \alpha(1+\gamma)Z(\omega,\eta)\\ &\quad -\alpha N(\omega,\eta)\end{aligned} \tag{4}$$

From (4) the Double Fourier Transform of $z_k(t)$, $Z(\omega, \eta)$, can be expressed as a function of the Double Fourier Transform of $n(z_k)$, $N(\omega, \eta)$:

$$Z(\omega,\eta) = -L(\omega,\eta)N(\omega,\eta) \tag{5}$$

with:

$$\begin{aligned}L(\omega,\eta) &= \frac{\alpha(-\omega^2 + j\omega(1+\gamma)+\beta)}{D(\omega,\eta)}\\ D(\omega,\eta) &= -j\omega^3 - \omega^2(1+\alpha)(1+\gamma)\\ &\quad +j\omega[\beta + 2\alpha\gamma(1-\cos\eta)] + \alpha\beta(1+\gamma)\end{aligned} \tag{6}$$

Therefore the dynamical system (1) can be represented in the Lur'e form shown in Fig. 2. Note that for $j\omega = s$ and $\gamma = 0$ (i.e no coupling) the transfer function $L(\omega, \eta)$ coincides exactly with that reported in formula (12) of [11].

The spectral approach that we propose is based on the Lur'e representation of Fig. 2 and extends the technique developed in [10] to systems that are both time and space dependent. It consists of two fundamental steps: the first one is the prediction of the existence of a periodic wave by means of a suitable extension of the describing function technique; the second is the evaluation of the distortion index which provides, according to [10], an approximate tool for determining whether chaos occurs.

In order to investigate the existence of a periodic wave, we represent the state $z_k(t)$ in the following approximate way:

$$z_{k0}(t) = A[\sin(\omega_0 t + k\eta_0) - \sin(\omega_0 t - k\eta_0)] \tag{7}$$

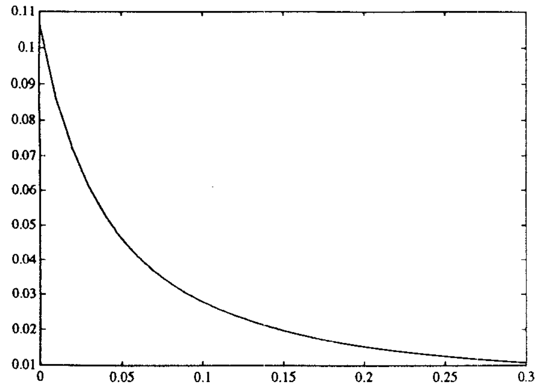

Figure 3: *Square of the distortion index* Δ^2, *versus* γ

where A and ω_0 are parameters to be determined whereas η_0 is assumed to be equal to the frequency of the first spatial harmonic, i.e. $\eta_0 = 2\pi/M$. It is easily verified that the choice of $z_{k0}(t)$ satisfies the boundary conditions $z_0(t) = z_M(t) = 0$ as required. Note that different boundary conditions can be satisfied by simply changing the approximate function $z_{k0}(t)$.

According to (2), the output of the nonlinear block of the Lur'e system $n[z_k(t)]$, results to be expressed by:

$$\begin{aligned} n[z_{k0}(t)] &= [-\tfrac{m}{G}A + \tfrac{9k}{4G}A^3] \\ &\cdot[\sin(\omega_0 t + k\eta_0) - \sin(\omega_0 t - k\eta_0)] \\ &-\tfrac{k}{4G}A^3[\sin 3(\omega_0 t + k\eta_0) - \sin 3(\omega_0 t - k\eta_0)] \\ &-\tfrac{3k}{4G}A^3[\sin(\omega_0 t + 3k\eta_0) - \sin(\omega_0 t - 3k\eta_0)] \\ &+\tfrac{3k}{4G}A^3[\sin(3\omega_0 t + k\eta_0) - \sin(3\omega_0 t - k\eta_0)] \end{aligned} \quad (8)$$

By neglecting both spatial and time higher order harmonics, $n[z_{k0}(t)]$ can be approximated by:

$$n[z_{k0}(t)] \approx [-\frac{m}{G}A + \frac{9k}{4G}A^3] \cdot[\sin(\omega_0 t + k\eta_0) - \sin(\omega_0 t - k\eta_0)] \quad (9)$$

Now, according to the Lur'e structure of the system of Fig. 2, in order to have a periodic wave the following pair of constraints in the two unknowns ω and A have to be satisfied:

$$\mathrm{Im}[L(\omega, \eta_0)] = 0 \quad (10)$$

$$[-\frac{m}{G} + \frac{9k}{4G}A^2]L(\omega, \eta_0) + 1 = 0 \quad (11)$$

From (10) it is possible to determine analytically the time oscillation frequency; there are two values of ω satisfying (10), but only one of them is compatible with a real value

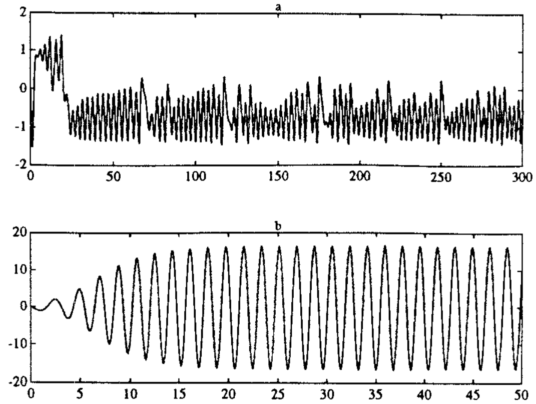

Figure 4: *Time-domain simulation of* $z_6(t)$ *versus* t*: (a)* $\gamma = 0.1$, *(b)* $\gamma = 0.2$

of A in (11):

$$\omega_0 = \sqrt{\beta + \delta - \frac{(1+\alpha)(1+\gamma)^2}{2} + \sigma} \quad (12)$$

$$\delta = \alpha\gamma(1 - \cos(\eta)) \quad (13)$$

$$\sigma = \sqrt{\left[\frac{(1+\alpha)(1+\gamma)^2}{2} - \delta\right]^2 - \beta(1+\gamma)^2} \quad (14)$$

Finally by (11) the approximate amplitude of the periodic wave is derived:

$$|A| = \sqrt{\frac{4G}{9k}\left[L(\omega_0, \eta_0) - \frac{m}{G}\right]} \quad (15)$$

This completes the prediction of the characteristic parameters of the periodic wave (7).

The principal limitation of the above procedure derives from the fact that the higher order harmonics have been neglected; in order to evaluate such an approximation we compute the distortion index, defined, according to [10], as:

$$\Delta = \frac{\|\tilde{z}_k(t) - z_{k0}(t)\|}{\|z_{k0}(t)\|} \quad (16)$$

where $\tilde{z}_k(t)$ represents the output of the open Lur'e system of Fig. 2 when the input is $z_{k0}(t)$ and the norm of a function $f_k(t)$ is defined as:

$$\|f_k(t)\| = \frac{1}{MT}\sum_{k=0}^{M-1}\int_0^T |f_k(t)|^2 dt \quad (17)$$

After some algebraic manipulation the explicit espression of Δ^2 results to be:

$$\Delta^2 = \frac{k^2A^4(|L(3\omega_0, 3\eta_0|^2 + 9|L(3\omega_0, \eta_0|^2 + 9|L(\omega_0, 3\eta_0|^2)}{16G^2|L(\omega_0, \eta_0|^2} \quad (18)$$

A small value of the distortion index indicates a low-pass filtering both in time and in space: in this case the existence of the predicted periodic wave is reliable. In [10] it is shown that for time-dependent systems there is an interval of values of the distortion index (medium filtering condition) corresponding to the existence of noisy limit cycles, that represent a strong indication of chaos. We will show that the distortion index plays a similar role also for the system under analysis; in particular if it crosses a threshold value a noisy periodic wave occurs, that can be interpreted as the occurrence of spatio-temporal chaos.

In order to show that, we assume $M = 10$, $\alpha = 19/2$, $\beta = 100/7$ and we only vary the parameter γ representing the coupling between the cells. With $\gamma = 0$ the cells are not coupled and each circuits works in a chaotic region (see [11]).

The plot of the square of the distortion index Δ^2 versus γ is reported in Fig. 3, while the simulation of the chain of Chua's circuits is shown in Fig. 4 for two characteristic values of γ.

The simulation shows that chaos occurs for γ less or equal to 0.1, and it disappears by increasing γ (see Fig. 4(b)). By looking at the plot of Fig. 3 it is seen that there exists a threshold value $\Delta_*^2 \approx 0.03$ (corresponding to $\gamma = 0.1$), such that for $\Delta^2 > \Delta_*^2$ the system exhibits spatio-temporal chaos, whereas for $\Delta^2 < \Delta_*^2$ the system converges towards a periodic wave. By performing other simulations (with different parameters α and β) we have found that the value of γ corresponding to the occurrence of chaos does change, but the threshold value Δ_*^2 remains approximately constant. It turns out that by the computation of the distortion index we have a simple tool for predicting chaos, that is of importance because only a few analytical tools are available for studying the dynamics high-dimensional systems.

IV. Conclusions

We have proposed a spectral technique for predicting chaos in a one-dimensional array of Chua's circuits; such a technique represents the extension of the method proposed by Genesio and Tesi in [10] to systems that have both time and space dependence.

The method gives an approximate tool for detecting chaos by simply performing some algebraic computations; its accuracy has been confirmed by time-simulation.

Future work will concern the complete study of the bifurcation phenomena in the space of the parameters α and β of the chain of Chua's circuits (that can be carried out analytically if the nonlinearity is approximated by a suitable polynomial).

Finally we remark that the method is suitable for all networks made of elementary identical nonlinear cells (then for example for Cellular Neural Networks exhibiting a complex dynamics).

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