Investigation of Chaos in Large Arrays of Chua's Circuits via a Spectral Technique

Marco Gilli, *Member, IEEE*

***Abstract*—A spectral technique is proposed for studying and predicting chaos in a one-dimensional array of Chua's circuits. By use of a suitable double transform, the network is reduced to a scalar Lur'e system to which the describing function technique is applied for discovering the existence and the characteristics of periodic waves. Finally, by the computation of the distortion index, an approximate tool is given for detecting the occurrence of chaos.**

I. Introduction

In recent years much interest has been devoted to the study of the dynamics of networks composed of elementary cells that exhibit a chaotic behavior (see [1], [3], [4], and [7]–[10]).

In particular in [3] and [4], a classification of the dynamic phenomena occurring in arrays of chaotic oscillators has been established. In the electrical engineering community, many researchers concentrated on the study of large arrays made of Chua's circuits (that is, a simple and robust example of chaotic oscillator [5], [6]); it has been shown that such arrays may model the propagation failure phenomenon [8] and may have application in image processing [9], [10].

One of the most interesting behaviors which can be observed in a one-dimensional array of Chua's circuits is the spatio-temporal chaos [7]. Due to the high dimension of such networks, only a few analytical tools are available for their study [2].

In this paper we investigate the dynamic behavior of a chain of Chua's circuits by use of a spectral technique which represents the extension of the technique introduced by Genesio and Tesi in [11] to systems that have both time and space dependence.

The method is based on the fact that all the cells are identical, and therefore, by introducing a suitable double transform, the network can be reduced to a scalar Lur'e system (see Fig. 2). By analyzing such a system, the existence of equilibrium points and their stability is investigated; then by use of an extension of the describing function technique (see [11]), the existence of periodic waves can be predicted. Finally by computing the distortion index, an approximate tool is developed for detecting the occurrence of chaos. The accuracy of the proposed technique has been confirmed by means of time simulation.

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The author is with the Cattedra di Elettrotecnica, Dipartimento di Elettronica, Politecnico di Torino, I-10129, Torino, Italy.

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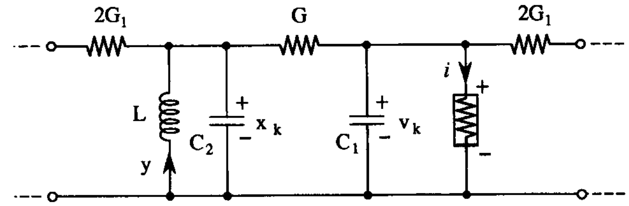

Fig. 1. The fundamental cell of the chain.

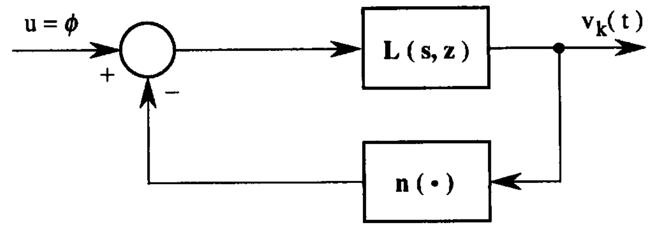

Fig. 2. Chain of Chua's circuits as a Lur'e system.

We remark that the advantage of the method is that it allows the prediction of chaos by means of simple algebraic computations and without performing any simulations (which, for large arrays of nonlinear circuits, are rather time consuming); moreover the method is not only applicable to arrays of Chua's circuits but to any network composed by identical elementary cells.

II. Chain Interconnections of Chua's Circuits

We consider the structure obtained by interconnecting a large number of classical Chua's circuits [5]: each cell of the chain (shown in Fig. 1), composed by two capacitors C_1 and C_2, an inductor L, a conductance G, and a Chua's diode ([5]) is coupled to the other cells by means of a conductance G_1.

To simplify the analysis, we assume that the chain is made by infinitely many cells; we will show by means of the simulation that if the number of cells is large enough (greater than 20), then an infinite array of Chua's circuits represents a good approximation of a finite one.

The dynamics of the kth cell of Fig. 1 after the scaling transformation $\tau = tG/C_2$ and $w_k = y_k/G$ can be described by the following set of equations

$$\begin{aligned}\frac{dx_k}{d\tau} &= -(x_k - v_k) + w_k - \gamma(x_k - v_{k-1})\\ \frac{dw_k}{d\tau} &= -\beta x_k \\ \frac{dv_k}{d\tau} &= -\alpha(v_k - x_k) - \alpha n(v_k) - \alpha\gamma(v_k - x_{k+1})\end{aligned} \tag{1}$$

where $\alpha = \dfrac{C_2}{C_1}$, $\beta = \dfrac{C_2}{LG^2}$, $\gamma = \dfrac{G_1}{G}$, $n(v_k) = \dfrac{1}{G}i(v_k)$.

We assume, according to [12], that the nonlinearity of the Chua's diode can be approximated by

$$n(v_k) = -\frac{m}{G}v_k + \frac{k}{G}v_k^3. \tag{2}$$

By use of (1) and (2), it is verified that the origin is an equilibrium point of the system for each value of γ. Moreover if the cells are not coupled (i.e., $\gamma = 0$) and $m/G > 1$, the kth Chua's circuit exhibits two other equilibria located at

$$(\overline{x}_k, \overline{w}_k, \overline{v}_k) = \left(0, \mp\sqrt{\frac{G}{k}\left(\frac{m}{G} - 1\right)}, \pm\sqrt{\frac{G}{k}\left(\frac{m}{G} - 1\right)}\right).$$

If the cells are coupled and $m/G > 1 + \gamma$, each circuit of the chain presents two equilibria whose coordinates $\overline{x}_k = 0$ and

$$\overline{v}_k = \pm\sqrt{\frac{G}{k}\left(\frac{m}{G} - (1+\gamma)\right)} \tag{3}$$

do not depend on the other cells, whereas the coordinate $\overline{w}_k$ does.

We fix the parameters of (2) to the values $m = 4/5$, $k = 2/45$, $G = 7/10$, which ensure that the kth cell of the chain, if not coupled, has three equilibria located at $\overline{v}_k = -1.5$, $\overline{v}_k = 0$, $\overline{v}_k = 1.5$ (see [12]). In the next section, the time variable τ appearing in (1) will be denoted again with t.

III. The Spectral Method

For studying the above dynamical system we propose a spectral technique based on the introduction of the double transform $F(s,z)$ of functions $f_k(t)$ discrete in space and continuous in time

$$F(s,z) = \sum_{k=-\infty}^{k=\infty} z^{-k} \int_{-\infty}^{\infty} f_k(t)\exp(-st)dt. \tag{4}$$

By applying the double transform defined above to the set of (1), the following relations are obtained

$$\begin{aligned} sX(s,z) &= -(1+\gamma)X(s,z) + W(s,z) \\ &\quad + (1+\gamma z^{-1})V(s,z) \\ sW(s,z) &= -\beta X(s,z) \\ sV(s,z) &= \alpha(1+\gamma z)X(s,z) \\ &\quad - \alpha(1+\gamma)V(s,z) - \alpha N(s,z). \end{aligned} \tag{5}$$

From (5) the double transform of $v_k(t)$, $V(s,z)$, can be expressed as a function of the double Fourier transform of $n(v_k)$, $N(s,z)$

$$V(s,z) = -L(s,z)N(s,z) \tag{6}$$

with (7) (shown at the bottom of the page). Note that for $\gamma = 0$ (i.e., no coupling) the transfer function $L(s,z)$ coincides exactly with that reported in [12, formula (12)].

Therefore the dynamical system (1) can be represented in the Lur'e form shown in Fig. 2, where the linear part is the transfer function $L(s,z)$ computed above, the nonlinear one is the function $n(\cdot)$ described in (2), and the output is the voltage $v_k(t)$ across the capacitor C_1, of the kth cell.

We will show that by use of the Lur'e scheme of Fig. 2 the existence and the stability of equilibrium points, periodic waves (i.e., spatio-temporal limit cycles) and even the occurrence of chaos can be investigated.

For what concerns the equilibrium points we have already pointed out that their coordinate $\overline{v}_k$ depends only on the nonlinearity of each cell and not on the state of the others; therefore $\overline{v}_k$ may be also computed as a solution of the Lur'e system, constant both in time and in space, i.e., by setting $s = 0$ and $z = 1$. By doing this, one obtains from Fig. 2:

$$n(\overline{v}_k)L(0,1) + \overline{v}_k = 0. \tag{8}$$

It is worth noting that besides the origin, the solutions of the above equation are exactly those reported in (3).

As far as the stability is concerned we concentrate on equilibrium points, whose coordinates $\overline{v}_k$ are given by (3); their stability can be studied by computing the zeros of the following equation

$$\left.\frac{dn(v_k)}{dv_k}\right|_{v_k=\overline{v}_k} L(s,1) + 1 = 0. \tag{9}$$

In particular, for each $\gamma < m/G - 1$, it is possible to determine in the plane of the parameters (α, β) the Hopf-curve, i.e., the curve where (9) has one negative and two purely imaginary roots; such a curve delimits the region of stability of the equilibrium points; its expression results to be (10) (shown at the bottom of the page).

For studying the occurrence of limit cycles and of chaotic attractors, we propose a spectral method which extends the technique developed in [11] to systems which have both temporal and spatial dependence. Such a method is based on the prediction of the existence of a symmetric periodic wave by means of a suitable extension of the describing function technique and then on the evaluation of the distortion index which provides, according to [11], an approximate tool for determining whether chaos occurs.

$$L(s,z) = \frac{\alpha(s^2 + s(1+\gamma) + \beta)}{s^3 + s^2(1+\alpha)(1+\gamma) + s[\beta + 2\alpha\gamma - \alpha\gamma(z + z^{-1})] + \alpha\beta(1+\gamma)} \tag{7}$$

$$\beta = -\frac{\alpha(2m/G - 3(1+\gamma))(1+\gamma)[(2m/G - 3(1+\gamma))\alpha + (1+\gamma)(1+\alpha)]}{1+\gamma} \tag{10}$$

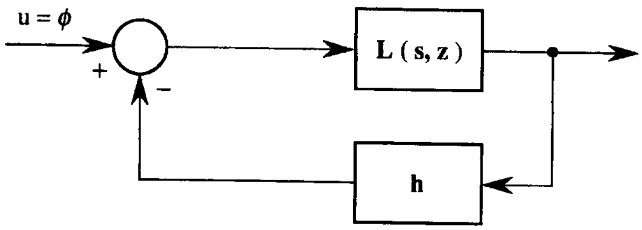

Fig. 3. Lur'e system, linearized via the describing function technique.

The extension of the describing function technique to spatio-temporal system consists of the following two steps:

- The state $v_k(t)$ is approximated by a periodic wave in the following way:
$$v_{k0}(t) = A\sin(\omega_0 t + k\eta_0) \tag{11}$$
where A is the amplitude and ω_0 and η_0 represent the temporal and the spatial frequency, respectively.
- According to [13], the nonlinear block is replaced by a simple constant gain h, having the same input (11), which minimizes the mean squared value of the error between the output from the nonlinearity and that from the gain itself. Note that h represents the proportionality factor between the input and the output obtained by substituting $v_{k0}(t)$ in (2) and by neglecting higher order harmonics.

By means of the above two steps, the nonlinear system of Fig. 2 is reduced to the linear one of Fig. 3 that is completely described by its dispersion equation in the spectral domain (s, z)
$$D(s,z) = 1 + hL(s,z) = 0. \tag{12}$$

From the above equation, the gain h and the temporal and the spatial frequency of the periodic wave may be predicted analytically; then by using (2) the approximate amplitude A can be derived as well.

In fact, so that the system of Fig. 3 has a solution without excitation, it is required that the dispersion equation, solved with respect to z, admits of a double root (see [14, p. 41]); moreover in order that the system supports the periodic wave (11) such a double root must occur at $s = j\omega_0$ and $z = \exp(j\eta_0)$.

It is easily derived that the above constraint may be imposed without finding explicitly the roots of (12), but simply by solving the following set of three equations in the three unknowns h, ω_0, η_0
$$\text{Im}[D(j\omega_0, \exp(j\eta_0))] = 0 \tag{13}$$
$$\text{Re}[D(j\omega_0, \exp(j\eta_0))] = 0 \tag{14}$$
$$\frac{dD(j\omega_0, \exp(j\eta_0))}{d\eta_0} = 0. \tag{15}$$
After a little algebra it is derived that (15) yields $\eta_0 = 0$, which physically means that the cells oscillate in phase.

The temporal frequency ω_0 is derived by (13) (note that there are two positive values of ω satisfying (13), but only one of them is compatible with a value of h, giving rise to a real value of A and to a stable limit cycle, according to the Loeb criterion [13])
$$\omega_0 = \sqrt{\beta - \frac{(1+\alpha)(1+\gamma)^2}{2} + \sqrt{\left[\frac{(1+\alpha)(1+\gamma)^2}{2}\right]^2 - \beta(1+\gamma)^2}}. \tag{16}$$
Finally by (14) the variable gain h turns out to be
$$h = -\frac{1}{\text{Re}[L(j\omega_0, 1)]}. \tag{17}$$

Now to determine the approximate amplitude of the periodic wave A, let us compute explicitly the output of the nonlinear block of the Lur'e system $n[v_k(t)]$, when the input is (11). We obtain
$$n[v_{k0}(t)] = [-\frac{m}{G}A + \frac{3k}{4G}A^3]\sin(\omega_0 t + k\eta_0) - \frac{k}{4G}A^3 \sin 3(\omega_0 t + k\eta_0). \tag{18}$$

By neglecting both spatial and temporal higher order harmonics, $n[v_{k0}(t)]$ can be approximated by
$$n[v_{k0}(t)] \approx [-\frac{m}{G}A + \frac{3k}{4G}A^3]\sin(\omega_0 t + k\eta_0) \tag{19}$$
and therefore the variable gain h, as a function of A is expressed by
$$h = -\frac{m}{G} + \frac{3k}{4G}A^2. \tag{20}$$
By combining (17) and (20), the amplitude A of the periodic wave comes out to be
$$|A| = \sqrt{\frac{4G}{3k}\left[\frac{m}{G} - \frac{1}{\text{Re}[L(j\omega_0, 1)]}\right]}. \tag{21}$$

This completes the prediction of the characteristic parameters of the periodic wave (11).

The principal limitation of the above procedure derives from the fact that the higher order harmonics have been neglected; to evaluate such an approximation we compute the distortion index, defined, according to [11], as
$$\Delta = \frac{\|\tilde{v}_k(t) - v_{k0}(t)\|_2}{\|v_{k0}(t)\|_2} \tag{22}$$
where $\tilde{v}_k(t)$ represents the output of the open Lur'e system of Fig. 2 when the input is $v_{k0}(t)$ and $\|\ \|_2$ is the L_2 norm over the period $2\pi/\omega_0$.

After some algebraic manipulations the explicit espression of Δ comes out to be
$$\Delta = \frac{kA^2|L(3j\omega_0, 1)|}{4G|L(j\omega_0, 1)|}. \tag{23}$$

A small value of the distortion index indicates a low-pass filtering both in time and in space: in this case the existence of the predicted periodic wave is reliable. In [11], it is shown that for time-dependent systems there is an interval of values of the distortion index (medium filtering condition), corresponding to the existence of noisy limit cycles, which represents a

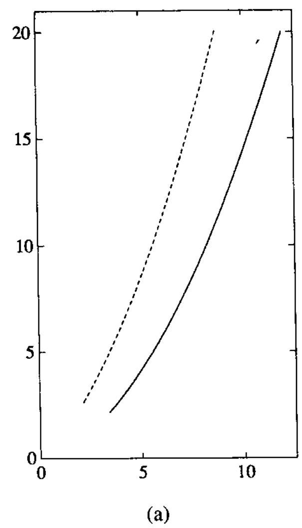

Fig. 4. Hopf curve (dashed) and distortion curve $\Delta = 0.03$ (solid); the parameters α and β are represented on the horizontal and vertical axis, respectively, (a) $\gamma = 0.05$, (b) $\gamma = 0.1$.

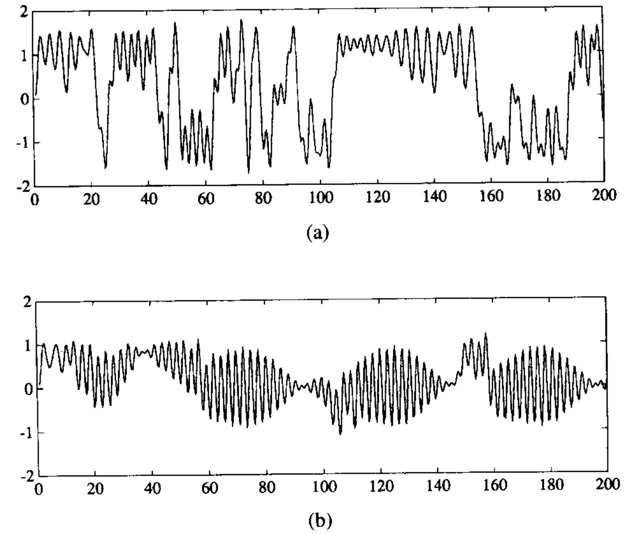

Fig. 6. Time-domain simulation of $v_{12}(t)$ for $\alpha = 6$, $\beta = 8$; (a) $\gamma = 0.05$, (b) $\gamma = 0.1$.

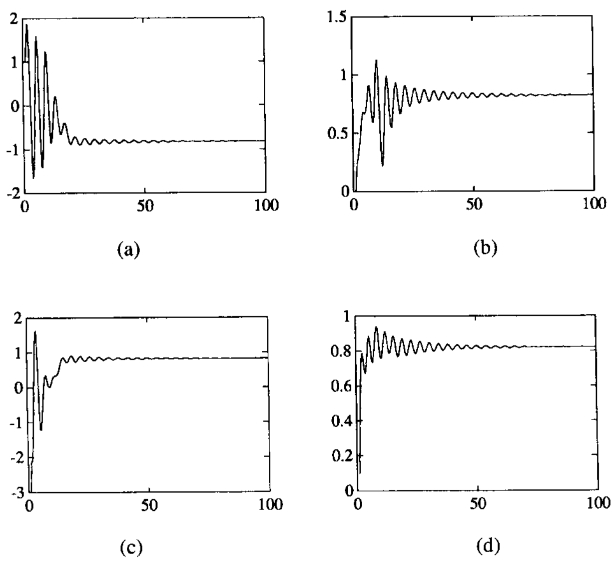

Fig. 5. Time-domain simulation for $\alpha = 5$, $\beta = 8$ and $\gamma = 0.1$ of (a) $v_9(t)$, (b) $v_{10}(t)$, (c) $v_{11}(t)$, and (d) $v_{12}(t)$ versus t.

strong indication of chaos. We will show that the distortion index plays a similar role also for the system under analysis, in particular if it crosses a threshold value (corresponding to $\Delta = 0.03$), a noisy periodic wave occurs, that can be interpreted as the occurrence of spatio-temporal chaos.

We have reported in the plane (α, β) the Hopf curve given by (10) and the distortion curve corresponding to $\Delta = 0.03$, for two values of the coupling parameter $\gamma = 0.05$ and $\gamma = 0.1$ (see Fig. 4(a) and (b)). We expect that chaos occurs in the region within the two curves where there are no stable equilibria and the predicted periodic wave is noisy, owing to the fact that $\Delta > 0.03$. To verify such a conjecture we have simulated a chain of Chua's circuits composed by 22 cells for $\beta = 8$ and different values of α lying inside and outside the above region. The results of the simulations, that have been performed by using the routine D02CAF from the NAG library, are the following:

- For $\gamma = 0.1$ and $\alpha = 5$, i.e., for a point which is close but to the left of the Hopf curve, the system, as expected, converges toward an equilibrium point [whose coordinates are given by (3)] after a rather long transient (see Fig. 5).
- For $\gamma = 0.1$ and $\alpha = 6$ and for $\gamma = 0.05$ and $\alpha = 6$, i.e., for points lying within the two curves (see Fig. 4(a) and (b)), chaos actually occurs (see Fig. 6); note that the simulations of Fig. 6 are shown till $t = 200$ but we have verified that for large values of t the system does not converge toward a periodic attractor.
- For $\gamma = 0.1$ and $\alpha = 7.6$, i.e., for a point lying to the right of the distortion curve, the system exhibits as expected a periodic wave whose characteristics are exactly those predicted by the describing function technique developed above. In particular, it is seen from Fig. 7 that the cells in the middle of the chain, even if they start from different initial conditions, in the steady state oscillate in phase, as predicted by $\eta_0 = 0$. Moreover the temporal period (approximately 2.4), is very close to that predicted by (16), i.e., $2\pi/\omega_0 = 2.38$; the amplitude of the oscillations of Fig. 7 is approximately 4.75, whereas from (21) we have $A = 4.62$, that represents a good prediction.

Extensive simulations of the system, for different values of α and β, have confirmed that in the region between the Hopf and the distortion curve we have a high probability to observe chaos. We would like to point out, however, that our method is approximate and therefore in a very few cases it might fail in the chaos prediction, especially if the parameters α and β are very close to the Hopf or to the distortion curve. Nevertheless all the simulations have shown that by fixing β and moving α between the above two curves, chaos is encountered. It turns out that the spectral method we have proposed provides a simple tool for predicting chaos, which is of importance

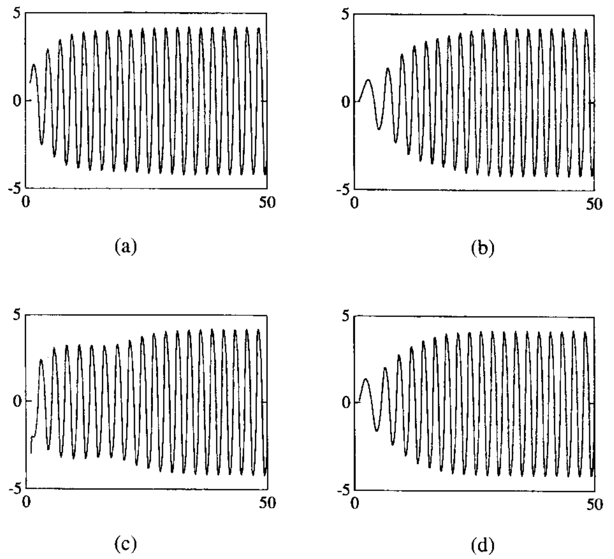

Fig. 7. Time-domain simulation for $\alpha = 7.6$, $\beta = 8$ and $\gamma = 0.1$ of (a) $v_9(t)$, (b) $v_{10}(t)$, (c) $v_{11}(t)$, and (d) $v_{12}(t)$ versus t.

because only a few analytical tools are available for studying the dynamics of high-dimensional systems.

IV. Conclusions

We have proposed a spectral technique for predicting chaos in a one-dimensional array of Chua's circuits; such a technique represents the extension of the method proposed by Genesio and Tesi in [11] to systems that have both time and space dependence.

By use of a suitable double Laplace transform, we have represented the network as a scalar Lur'e system in the spectral domain; then by analyzing such a system and by means of a suitable extension of the describing function technique, we have developed an approximate and simple tool for detecting chaos.

In particular for each value of the coupling parameter between the cells we have divided the space of the parameters α and β of the Chua's circuit into three regions, approximately corresponding to those where the system converges toward an equilibrium point, a periodic wave, or a chaotic attractor, respectively. The accuracy of the method has been confirmed by means of the time simulation of the system.

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Marco Gilli (M'93), for a photograph and biography, see this issue, p. 582.