ANALYSIS OF NONLINEAR DYNAMIC ARRAYS, THROUGH SPATIAL MODE DECOMPOSITION

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ABSTRACT

A method for the spectral analysis of the spatial modes of a 1D arrays of Chua's circuits is proposed. The analysis is based on two steps: (a) the state is represented in terms of a set of spatial eigenfunctions; (b) a set of nonlinear differential equations involving the coefficients of the eigenfunctions is derived; this set is completely equivalent to the original set of equations describing the circuit array. The spectral analysis allows to explain some spatio-temporal dynamic phenomena occurring in arrays of Chua's circuits. The technique can be extended to more complex 1D arrays and to 2D arrays of nonlinear cells.

1. INTRODUCTION

Dynamic arrays of nonlinear cells have found several applications in image processing and for modelling pattern formation and wave phenomena in physics, chemistry and biology [1]. In most applications the cells are identic and the interactions among the cells are described by space-invariant templates. Such arrays are dynamical systems described by a large set of nonlinear differential equations: therefore the study of their dynamics has been mainly carried out through extensive computer simulations.

An alternative way for studying the dynamic phenomena occurring in these networks are the spectral techniques. Spectral techniques consider the state of the network as a multiply function of time and of one or more discrete spatial coordinates. From this point of view the entire dynamical system can be described as a nonlinear spatio-temporal differential operator applied to the state of the network, with the addition of a set of boundary conditions (that describe the interactions among the cells located at the network boundary). The key step of spectral techniques is the representation of the state through a set of functions, that incorporate the boundary conditions and give rise to a considerable simplification of the spatio-temporal differential operator. If the state representation is an approximation (like for the harmonic balance technique, with a finite number of harmonics) then spectral techniques allows to predict accurately most of the dynamic phenomena occurring in nonlinear arrays [2]-[6]. If the state representation is exact, then the original set of differential equations is trasformed into an equivalent set of algebraic and/or differential equations, that in some cases might give more insight into the dynamic behavior of the network. In particular in [7] and [8] the state has been decomposed through the eigenfunctions of a suitable linearization of the spatio-temporal differential operator; through this decomposition several mechanisms of pattern formation have been easily explained.

In this paper we restrict our attention to one-dimensional arrays of Chua's circuits [9], that, with respect to other networks, exhibit a richer dynamic behavior. As a first step we represent the state through the eigenfunctions (modes) of the spatial part of the spatio-temporal nonlinear operator. Then we prove that this leads to a set of nonlinear differential equations, that describes exactly the network and is completely equivalent to the original one (i.e., it is neither a linear approximation, nor an approximation with a finite number of harmonics). Finally we show that the spatial eigenfunctions represent a frame that allows to explain in a simpler way the dynamics of the network.

2. ONE DIMENSIONAL ARRAY OF CHUA'S CIRCUITS

We consider a dynamic array composed by a finite number (N) of Chua's circuits and described by the set of normalized equations reported in [9]:

$$\begin{aligned}\dot{x}_k &= \alpha[y_k - x_k - n(x_k)] + d_1 x_{k-1} + d_2 x_{k+1} - 2d x_k\\ \dot{y}_k &= x_k - y_k + z_k\\ \dot{z}_k &= -\beta\, y_k\end{aligned}$$

The parameters α and β are defined in [10] whereas d_1 and d_2 represent the normalized coupling coefficients; $n(x_k)$ denotes the well known nonlinear memoryless resistance of the Chua's diode (see [10]). We assume that the boundary conditions are $x_0(t) = x_{N+1}(t) = 0$.

By eliminating in (2) $y_k(t)$ and $z_k(t)$ the following equation, expressed in term of sole variable $x_k(t)$, holds

$$\begin{aligned}L(D)[x_k(t)] + \alpha\, n[x_k(t)] - d_1 x_{k-1}(t) &\; - \\ d_2\, x_{k+1}(t) + 2d\, x_k(t) &= 0 \qquad (1)\end{aligned}$$

where D represents the first-order differential operator and

$$L(D) = \frac{D^3 + D^2(1+\alpha) + D\beta + \alpha\beta}{D^2 + D + \beta} \qquad (2)$$

By denoting with ζ^m the spatial operator defined as $\zeta^m(x_k(t)) = x_{k+m}(t)$, equation (1) can be rewritten in the following more compact form:

$$\mathcal{Q}[x_k(t)] = 0 \qquad (3)$$

where $\mathcal{Q}$ is a spatio-temporal nonlinear differential operator defined as:

$$\mathcal{Q}(\cdot) = L(D)(\cdot) + \alpha\, n(\cdot) - d_1\, \zeta^{-1}(\cdot) - d_2\, \zeta^1(\cdot) + 2d\, \zeta^0(\cdot) \qquad (4)$$

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For the sake of the simplicity we only study the case of reciprocal coupling (i.e., $d_1 = d_2 = d$). We denote with $\mathcal{Q}_s$ the spatial part of the operator $\mathcal{Q}$ (i.e., $\mathcal{Q}_s = d\ (2\zeta^0 - \zeta^{-1} - \zeta^1)$), that is linear. Since we have assumed zero boundary conditions, it is easily derived that the operator $\mathcal{Q}_s$ admits of the following complete set $\mathcal{F}_k$ of eigenfunctions (i.e., spatial modes that satisfy the boundary conditions):

$$\mathcal{F}_k = \left\{ \sin\left(\frac{m\pi k}{N+1}\right), \quad 1 \le m \le N \right\} \tag{5}$$

In fact, each one of the above eigenfunctions vanishes for $k = 0$ and $k = N + 1$ and moreover:

$$d\ (2\zeta^0 - \zeta^{-1} - \zeta^1)\ \sin\left(\frac{m\pi k}{N+1}\right) = \lambda_m \sin\left(\frac{m\pi k}{N+1}\right)$$
$$\lambda_m = d\ \sin^2\left(\frac{m\pi}{N+1}\right) \tag{6}$$

where λ_m is the eigenvalue corresponding to the eigenfunction $\sin\left(\frac{m\pi k}{N+1}\right)$.

The eigenfunctions F_k are also orthogonal, according to the following scalar product:

$$\frac{2}{N+1}\sum_{k=1}^{N} \sin\left(\frac{m_1\pi k}{N+1}\right)\sin\left(\frac{m_2\pi k}{N+1}\right) = \delta_{m_1,m_2} \tag{7}$$

where δ denotes the Kronecker delta operator. Since the set of eigenfunctions F_k is complete, then the state $x_k(t)$ can be represented in the following form:

$$x_k(t) = \sum_{m=1}^{N} X_m(t)\sin\left(\frac{m\pi k}{N+1}\right) \tag{8}$$

where the $X_m(t)$ are suitable coefficients, that take into account the time dependence of $x_k(t)$.

We will show that, by substituting expression (8) for $x_k(t)$ in (3), a set of nonlinear differential equations involving the terms $X_m(t)$ equivalent to (3) can be obtained. In fact it is possible to prove that also the output of the memoryless nonlinear function $n(x_k(t))$ can be exactly represented through the set of eigenfunctions F_k. We have:

$$n(x_k(t)) = \sum_{m=1}^{N} N_m(t)\sin\left(\frac{m\pi k}{N+1}\right) \tag{9}$$

where the terms $N_m(t)$ depend only on the coefficients $X_m(t)$ of the state representation (8) and are defined as

$$N_m(t) = \hat{N}_m[X_1(t), X_2(t), \ldots X_N(t)] =$$
$$\frac{2}{N+1}\left[\sum_{k=1}^{N} n\left(\sum_{l=1}^{N} X_l(t)\sin\left(\frac{l\pi k}{N+1}\right)\right)\sin\left(\frac{m\pi k}{N+1}\right)\right] \tag{10}$$

By substituting expressions (8) and (10) in (3) and by use of (6) we obtain

$$L(D)\left[\sum_{m=1}^{N} X_m(t)\sin\left(\frac{m\pi k}{N+1}\right)\right] +$$
$$\alpha\sum_{m=1}^{N} \hat{N}_m[X_1(t), \ldots, X_N(t)]\sin\left(\frac{m\pi k}{N+1}\right) +$$
$$d\sum_{m=1}^{N} \sin^2\left(\frac{m\pi}{N+1}\right) X_m(t)\ \sin\left(\frac{m\pi k}{N+1}\right) = 0 \tag{11}$$

The above equation has the fundamental property that the space dependence is entirely concentrated in the eigenfunctions; therefore by taking the scalar product of both the sides of (11) with each one of the eigenfunctions $\sin\left(\frac{m\pi k}{N+1}\right)$ the following decomposition into N nonlinear differential equations holds:

$$L(D)X_m(t) + \alpha\ \hat{N}_m[X_1(t), \ldots, X_N(t)] +$$
$$d\sin^2\left(\frac{m\pi}{N+1}\right) X_m(t) = 0 \qquad 1 \le m \le N \tag{12}$$

The set of equations above presents the following properties:

- It is completely equivalent to the original set of equations, expressed through the spatio-temporal nonlinear operator $\mathcal{Q}$ in (3) and (4). This means that by computing $X_m(t)$ via the solution of set (12), it is possibile to determine exactly the state $x_k(t)$ by using (8).
- It allows to study the time evolution of the spatial eigenfunctions (modes), i.e., of the coefficients $X_m(t)$. This is important for investigating all those phenomena (like pattern and wave formation) that can be better described in terms of spatial modes (see [7, 8]). The main advantage of our approach with respect to [7, 8] is that our formulation is valid in the whole state-space and not only in a suitable linear region as in [7, 8].

The set of equations (12) can be studied through different approximate methods, like for example harmonic balance, that allow to predict accurately the dynamics of each spatial mode. On the other hand the evolution of the spatial eigenfunctions can also be determined by the time simulation of (12) through numerical methods.

As an example we examine a chain of 12 Chua's circuits, described by the parameters $\alpha = 8$, $\beta = 15$ and $d = 0.17$; we suppose that the nonlinearity of Chua's diode can be approximated through the cubic function [11]

$$n(x_k) = -\frac{8}{7}x_k + \frac{4}{63}x_k^3 \tag{13}$$

It is shown in [6] and [9] that, for small values of d (like for example $d = 0.17$), after a transient the network exhibits clusters of at least two cells oscillating around ± 1. The form of the clusters depends on the initial conditions. We have considered two cases: a first case, called symmetric, where the initial conditions are chosen in such a way that cells $1-4$ and cells $9-12$ oscillate around $+1$, whereas cells $5-8$ oscillate around -1; a second case, called

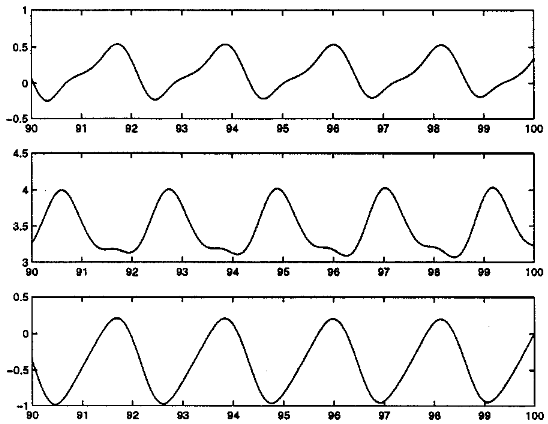

Figure 1: Time-waveforms of the coefficients $X_m(t)$ of the odd spatial eigenfunctions 1-3-5 in the symmetric case. They are ordered from the top to the bottom.

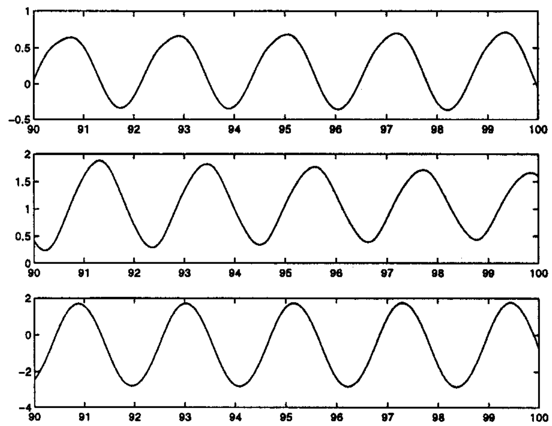

Figure 2: Time-waveforms of the coefficients $X_m(t)$ of the odd spatial eigenfunctions 7-9-11 in the symmetric case. They are ordered from the top to the bottom.

asymmetric, in which the initial conditions are chosen in such a way that cells $1-3$ and cells $9-12$ oscillate around $+1$, whereas cells $4-8$ oscillate around -1.

In both cases the cluster formation can be described in a effective way by computing the time evolution of the spatial modes. For what concerns the symmetric case it is easily verified that the even modes vanishes; the odd modes are reported in Figs. 1-2, after a transient long enough (i.e., from 90 to 100 normalized time units). It is seen from Fig. 1 that the average value of the waveform representing the third mode (i.e., $m = 3$) is much greater than those of the other modes. This is in agreement with the fact that a sequence of clusters of the type $(1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1)$ has been formed.

The waveforms representing the spatial modes in the asymmetric case are reported in Figs. 3-6: it is seen that all the modes are different from zero and that the amplitude of the third mode decreases. This is in agreement with the fact that a sequence of clusters of the type $(1, 1, 1, -1, -1, -1, -1, -1, 1, 1, 1, 1)$ has been formed.

The analysis of spatial modes through equation (12) allows to explain most of the mechanisms of cluster formation and also the spatio-temporal chaotic phenomena that occur for higher values of d ([6]).

3. CONCLUSION

We have proposed a method for the analysis of the spatial modes of a 1D arrays of Chua's circuits in the whole state space. We have verified that the analysis allows to explain some spatio-temporal dynamic phenomena occurring in the network. Moreover we point out that the technique can be extended to more complex 1D arrays and to 2D arrays of nonlinear cells.

4. REFERENCES

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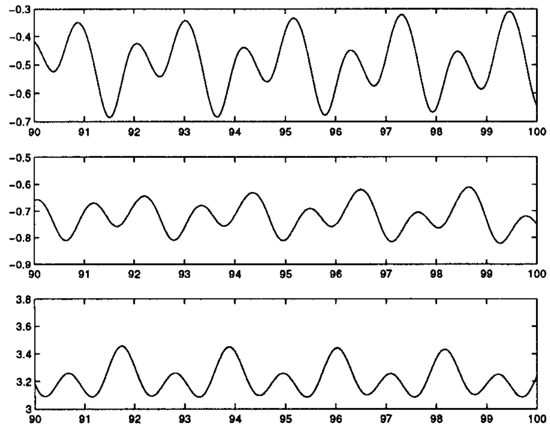

Figure 3: Time-waveforms of the coefficients $X_m(t)$ of the spatial eigenfunctions 1-3 in the asymmetric case. They are ordered from the top to the bottom.

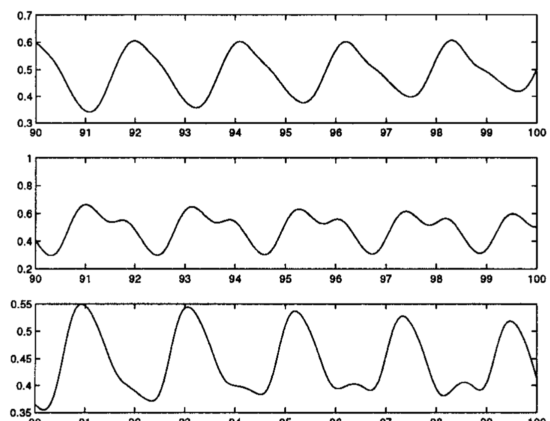

Figure 4: Time-waveforms of the coefficients $X_m(t)$ of the spatial eigenfunctions 4-6 in the asymmetric case. They are ordered from the top to the bottom.

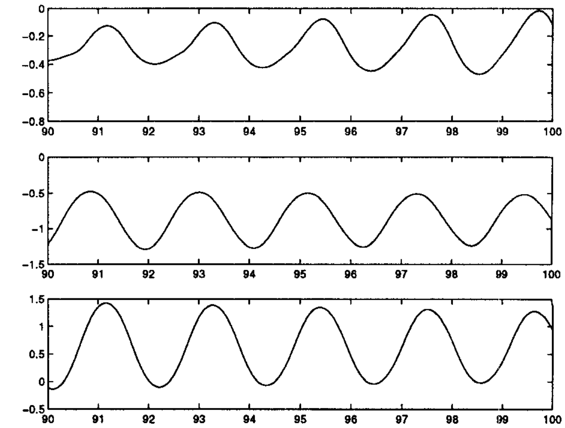

Figure 5: Time-waveforms of the coefficients $X_m(t)$ of the spatial eigenfunctions 7-9 in the asymmetric case. They are ordered from the top to the bottom.

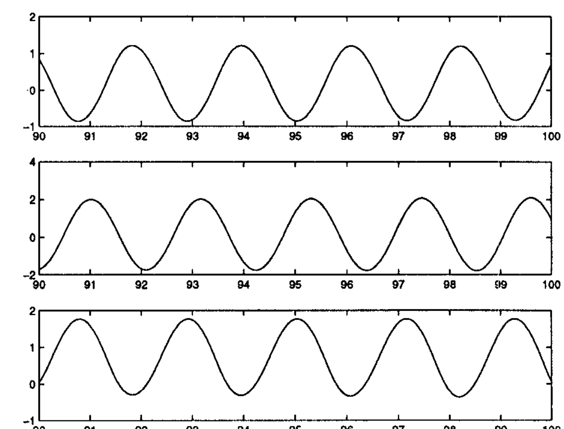

Figure 6: Time-waveforms of the coefficients $X_m(t)$ of the spatial eigenfunctions 10-12 in the asymmetric case. They are ordered from the top to the bottom.