PREDICTING PERIOD-DOUBLING BIFURCATIONS IN NONLINEAR TIME-DELAYED FEEDBACK SYSTEMS

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Abstract

A graphical approach is developed in this paper for detecting the period-doubling bifurcation emerging near the Hopf bifurcation point of a time-delayed feedback system. The new algorithm employs higher-order harmonic balance approximations (HBAs) for estimating the predicted periodic solutions of the system. Prediction of the period-doubling bifurcation is accomplished using a type of distortion index based on some information about the higher-order harmonics. The time-delayed Chua's circuit is used as an example for illustration.

1 Introduction

In the last two decades, there has been increasing interest in studying the coexistence of periodic and chaotic behaviors in nonlinear dynamical systems such as nonlinear circuits. Investigation of complex systems with multiple equilibria, multiple periodic solutions, and particularly chaotic behaviors has posed a real challenge to both analysts and engineers. In order to carry out qualitative analysis, it is preferable to have *simple* models that possessing such complicated but typical features, e.g., a simple time-delayed feedback system can have very complex dynamics. A notable feature of such systems is that they are described by hybrid types of dynamical equations such as difference-differential or functional differential equations, which are intrinsically complicated in terms of dynamical behaviors [1]. A representative circuit of this type is the time-delayed Chua's circuit (TDCC) [2,3]. The TDCC has typical chaotic dynamics, and can be used in many applications such as laboratory models for living neural networks, as a vehicle for bifurcation and chaos control, and for chaos synchronization with application in secure communications.

In this paper, we show how the analysis of simple time-delayed systems can be carried out by using engineering frequency domain techniques and harmonic balance approximations. First, the formulation of a general setting for time-delayed systems, studied in [4], is extended to difference-differential equations of the neutral type. A simple algorithm is then developed for predicting the appearance of period-doubling bifurcations emerging near the Hopf bifurcation point of the system. The proposed graphical analysis method along with higher-order HBAs is applied to predict the occurrence of the first period-doubling bifurcation, where the distortion index [3,5] is used for computation. The TDCC is finally simulated for testing and illustration of the new algorithm.

2 HBA for Time-Delayed Nonlinear Feedback Systems

Consider the following parametrized time-delayed nonlinear differential equation:

$$\begin{aligned} &\dot{x}(t-\tau) + A_0(\mu)\dot{x}(t) = A_1(\mu)x(t) \\ &\quad + A_2(\mu)x(t-\tau) + B(\mu)g\left(C(\mu)x(t-\tau);\mu\right), \qquad (1) \\ &y(t) = C(\mu)x(t), \end{aligned}$$

where A_0, A_1, and A_2 are $n \times n$ matrices, B is an $n \times r$ matrix, C is an $m \times n$ matrix, $\mu \in R$ is the main bifurcation parameter, $y \in R^m$ is the system output, $u = g[C(\mu)x(t-\tau);\mu] \in R^r$ is the system input; $g : R^m \to R^r$ is a smooth (C^{2p+1}, $p = 1, \ldots, q$) nonlinear function, and $\tau > 0$ is a time-delay constant.

The first step in applying the Graphical Hopf Method (GHM) is to recast the system into the Lur'e form by defin-

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ing (see Fig. 1.a)

$$\begin{aligned} e &= -y = -Cx, \\ G(s;\mu) &= C\left[s(\exp(-s\tau)+A_0) \right. \\ &\quad \left. -A_1 - A_2\exp(-s\tau)\right]^{-1}B, \\ u &= f[e(t-\tau);\mu] = g[Cx(t-\tau);\mu]. \end{aligned} \quad (2)$$

The equilibrium solution $\hat{e}$ is obtained from

$$G(0;\mu)f(e;\mu) = -e, \quad (3)$$

and its linearization around the feedback is given by

$$J\exp(-s\tau) = \left.\frac{\partial f}{\partial e}\right|_{e=\hat{e}} \exp(-s\tau). \quad (4)$$

The characteristic function can be written as

$$\det\ [\lambda I - G(s;\mu)J\exp(-s\tau)] = 0. \quad (5)$$

A Hopf bifurcation arises when one eigenvalue $\hat{\lambda}$ passes through the point $(-1+0i)$ for a given value of the main bifurcation parameter, μ_H, and when $s = i\omega_H$. The emerging periodic solutions are approximated by

$$e(t) = \hat{e} + \Re\left\{\sum_{k=0}^{2q} E^k \exp(ik\hat{\omega}t)\right\}, \quad (6)$$

where $\Re(\cdot)$ denotes the real part, $i=\sqrt{-1}$, $\hat{\omega}$ is the fundamental frequency, $q=1$ (second-order HBA), $q=2$ (fourth-order HBA), and so on, and E^k are complex numbers in the kth harmonic, satisfying

$$E^k\exp\left(ik\hat{\omega}(t-\tau)\right) = E_d^k\exp(ik\hat{\omega}t), \quad (7)$$

with subscript d standing for delayed quantities.

Suppose the main bifurcation parameter μ is fixed. The harmonic balance approximation method gives the following relation after equating the output of the linear plant $(-G(ik\omega;\mu)F_d^k)$ with the predicted periodic input (E^k) to the nonlinear path, i.e.,

$$E^k = -G(ik\omega;\mu)F_d^k, \quad k=0,1,2,\cdots,2q, \quad (8)$$

where F_d^k are the Fourier coefficients of $f(e;\mu)$, depending on the higher-order partial derivatives of $f(e;\mu)$ around the equilibrium. Assuming the standard filtering hypotheses of the linear transfer function, we can write

$$-G(ik\omega;\mu)F_d^k \approx 0, \quad k=2q+1,\cdots. \quad (9)$$

In order to have an indication of the error, we open the feedback path at $e(t)$ and input a test signal $e_a(t)$ (obtained, for instance, from a second-order HBA). Then we can define a measure for the error in filtering higher-harmonics by the linear transfer function (and in the truncated Taylor series). The measure is called the distortion index, Δ, and is defined by [3,5]

$$\Delta = \frac{\|e_a(t) - [-f(e_a(t-\tau))G(s;\mu)]\|}{\|e_a(t)\|} = \frac{\|e_a(t)-e_b(t)\|}{\|e_a(t)\|}, \quad (10)$$

where $\|\cdot\|$ is a suitable vector norm. Suppose that $q\to\infty$. If $e_a(t) = e_{true}(t)$ is a true periodic solution of (1), then it is a solution to the infinite harmonic equation set (matching $e_b(t)$ exactly) and we have $\Delta = 0$. So the distortion index can be used as a measure of the difference between the predicted limit cycle and the true periodic orbit.

The GHM comprises four sets of formulas, labeled L_q $(q=1,\cdots,4)$, that provides a solution to the harmonic balance set of $2q+1$ equations with increasing accuracy, through the evaluation of the expressions

$$\hat{\lambda}(i\omega;\mu) = -1 + \sum_{k=1}^{q}\xi_k(\omega)\theta^{2k} =: L_q, \quad (11)$$

where $\hat{\lambda}(i\omega;\mu)$ is the eigenvalue of $G(i\omega;\mu)J(\mu)$ closest to the critical point $(-1+i0)$, the complex numbers $\xi_k(i\omega)$ (used for calculating the amplitude and frequency of the periodic solution) are defined in [4], and θ is a measure of the amplitude of the (predicted) limit cycle. Then, in terms of the solutions $(\theta,\omega)_q$ $(q=1,\cdots,4)$, the values of E^k $(k=0,\cdots,2q)$ can be obtained. In this case, a version of the GHM for time-delayed systems is needed. We note that this can be obtained from the general formulas for systems without time delays [4] by simple corresponding modifications with the additional time-delayed term $\exp(-ik\omega\tau)$ (for details, see [6]).

In this paper, Eqn. (11) is solved exactly using the technique introduced in [7]. Although computationally this approach is more time-consuming as compared to the one derived in [4], it provides better approximations of the predicted limit cycles and, more importantly, gives significant insights for the period-doubling bifurcation phenomena if the amplitude θ is less than 1. To predict period-doubling bifurcations, the algorithm is designed with four steps:

Step 1: Find the exact solution pair $(\hat{\omega},\hat{\theta})$ of the amplitude locus $L_1(\omega,\theta,\mu)$ and frequency $\hat{\lambda}(i\omega,\mu)$, for a given value of μ in the proximity of the Hopf bifurcation point ($\omega=\omega_0$, $\theta=0$, $\mu=\mu_0$). Vary the value of μ to obtain the branch of periodic solutions emerging from the Hopf bifurcation point, until $\hat{\theta}=1$ and $\mu=\mu_{\theta=1}$.

Step 2: Find, if it exists, a second solution pair $(\hat{\omega}_{pd},\hat{\theta}_{pd})$ between $L_1(\cdot)$ ($L_2(\cdot)$, and so on) and $\hat{\lambda}(\cdot)$, provided that $\hat{\theta}_{pd} < 1$ in the same range of variation of the parameter μ, *i.e.*, $\mu_0 < \mu < \mu_{\theta=1}$ (or $\mu_{\theta=1} < \mu < \mu_0$), such that $\hat{\omega}_{pd} \approx \hat{\omega}/2$.

Step 3: Evaluate the distortion index Δ for $e_H(t)$ (Hopf solution) and $e_{pd}(t)$ (period-doubling solution). If the distortion index $\Delta_{pd}(\mu_{pd})$ (of the period-doubling solution) is smaller than the distortion index $\Delta_H(\mu_{pd})$ (of the Hopf solution), at least for one approximation L_q $(q=1,\cdots,4)$, then it is likely that a period-doubling bifurcation exists when μ_{pd} satisfies $\Delta_{pd}(\mu_{pd}) < \Delta_H(\mu_{pd})$ and $\Delta_{pd}(\mu_{pd}) = \min\Delta_{pd}(\mu)$.

3 A Circuit Example

The time-delayed Chua's circuit is (see [2,3])

$$\mu(1+\zeta)\frac{dx(\tau+1)}{d\tau}+\mu(1-\zeta)\frac{dx(\tau)}{d\tau}+x(\tau+1)$$
$$+x(\tau)+(1+\zeta)g(x(\tau+1))+(1-\zeta)g(x(\tau))=0,$$

where τ is the time variable normalized with respect to the time-lag (TL) delay, T; $g(\cdot)$ is the nonlinearity in the Chua's diode approximated by a cubic function $g(z)=(-mz+kz^3)/G$, with constants m and k, normalized with respect to the conductance G; $\zeta = ZG$, with the TL characteristic impedance Z, $\mu = C/GT$, and C is the capacitance. The transfer function $G(s;\mu)$ of the TDCC is

$$G(s;\mu)=\frac{1}{[\mu(1+\zeta)\exp(s)+\mu(1-\zeta)]s+\exp(s)+1},$$

and the Jacobian J of the nonlinear feedback function $f(e) =: g(x)$, after defining $e = -x$, is

$$J=-(1+\zeta)\left[\frac{m}{G}-\frac{3k\hat{e}^2}{G}\right]\exp(s)-(1-\zeta)\left[\frac{m}{G}-\frac{3k\hat{e}^2}{G}\right],$$

where $\hat{e}$ are the equilibrium solutions of the circuit, that is, $\hat{e}^{(1)}=0$ and $\hat{e}^{(2,3)}=\pm1.5$. We only analyze the system dynamics around the equilibrium solution $\hat{e}^{(3)}=-1.5$.

Under this framework and using the GHM, a Hopf bifurcation can be detected to occur at $\mu_0 = 0.192303$ and $\omega_0 = 2.349178$. The amplitude of the limit cycle emerging from the Hopf bifurcation is predicted using L_1 and L_2, with a comparison to the actual amplitude (computed by numerical integration), as shown in Figs. 2 and 3, respectively. Notice that while L_1-predictions are accurate only for those values of μ near μ_0 (see Fig. 2), L_2 provides an almost indistinguishable approximation for the limit cycle beyond the period-doubling bifurcation point (obtained by capturing the unstable periodic solution, see Fig. 3). In these figures, the values of the period-doubling bifurcation, predicted by using L_1 and L_2-approximations respectively, are indicated as μ_1 and μ_2 (in Figs. 2 and 3), respectively.

The GHM, in this case, also provides a second solution of the amplitude, $\theta_{pd} < 1$, and frequency, $\omega_{pd} \approx \frac{\omega_H}{2}$. In Fig. 4, plots of the distortion index of the predicted Hopf and period-doubling orbits are shown, for L_1 and L_2-approximations, respectively. In coincidence with Figs. 2 and 3, the distortion indexes Δ here indicate that the L_2-predictions are better than the L_1 ones, for both Hopf (L_{1H} and L_{2H}) and period-doubling orbits (L_{1pd} and L_{2pd}). It is observed that the Δ for period-doubling predictions (L_{1pd} and L_{2pd}) are smaller than that for Hopf predictions using L_1 in the range $[0.163, 0.17]$ of μ. This implies that period-doubling predictions are all close to the true period-doubling orbit. The values of Δ for predicting the Hopf bifurcation using L_2 are the smallest within that range; however it must be noted that these predicted results are close to the unstable Hopf limit cycle rather than the stable period-doubling orbit. Also, it has been observed in simulations that the minimum value of Δ for period-doubling predictions indicates a μ value near the critical value of $\mu_{pd} = 0.166$ which generates a period-two limit cycle.

As can be seen from Figs. 3 and 4, the outcome of L_2 gives a very good prediction for the Hopf limit cycle. Note that when a system undergoes period-doubling bifurcations the Hopf limit cycle at the frequency ω_H suffers an infinitesimal periodic deformation with frequency $\omega_{pd} = \frac{\omega_H}{2}$. In this case, the following numerical experience can be performed:

Step 1. Plot the curves $\Delta_{2H}(\mu)$ and $\Delta_\epsilon(\mu)$ for the range $\mu \in [0.1915, 0.163]$, where $\Delta_{2H}(\mu)$ is the distortion index of the L_2-approximation of the Hopf limit cycle, *i.e.*,

$$e_{2H}(t;\mu)=\hat{e}+\Re\left\{\sum_{k=0}^{4}E^k\exp(ik\omega t)\right\}$$

and $\Delta_\epsilon(\mu)$ is the distortion index of the test signal

$$e_\epsilon(t;\mu) = \hat{e}+\Re\left\{\sum_{k=0}^{2}E^k\exp(ik\omega t)\right\} +\epsilon\Re\left\{\sum_{k=1}^{2}E^k\exp(\frac{ik\omega t}{2})\right\},$$

where $\epsilon \leq 10^{-6}$ (a arbitrarily chosen small number). Here, the test signal has four harmonics instead of eight, due to implementation, *i.e.*, the software computes the distortion index using up to four harmonics for the input signal $e_a(t)$.

Step 2. Find the value of μ_g that satisfies $\Delta_\epsilon(\mu_g) < \Delta_{2H}(\mu_g)$ and $\Delta_\epsilon(\mu_g)$ is a minimum of $\Delta_\epsilon(\mu)$. The value μ_g can be considered as an empirical guess of the critical value μ_{pd} at which the period-doubling bifurcation occurs.

Figure 5 shows the result of this graphical analysis procedure, with $\epsilon = 10^{-9}$. The value $\mu_g = 0.166$ is equal to the critical value μ_{pd} for which a period-two limit cycle is observed in the simulation.

4 Conclusions

In this paper a graphical and computational approach has been developed for predicting period-doubling bifurcations near the Hopf bifurcation point of a nonlinear system. When applied to the time-delayed Chua's circuit, the predictions have shown to be correct and accurate. This method is convenient to use since all needed analytic formulas, computational tools, and computer-graphic software are available and well-tested.

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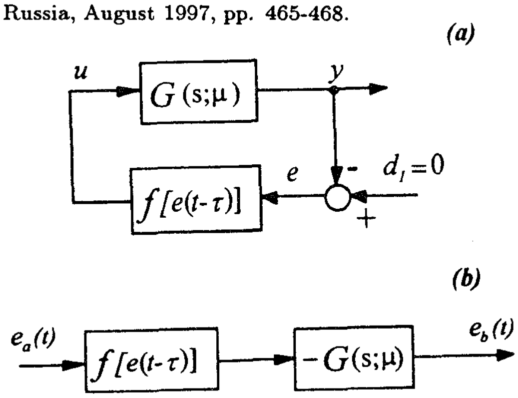

Figure 1: a) The nonlinear feedback system. b) The open loop feedback system.

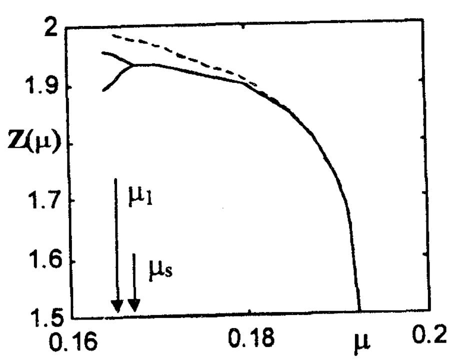

Figure 2: Simulation (solid line) and L_1 approximation (dashed line) of the periodic solution branch showing the prediction of period-doubling bifurcation (μ_1) with respect to the real value (μ_s).

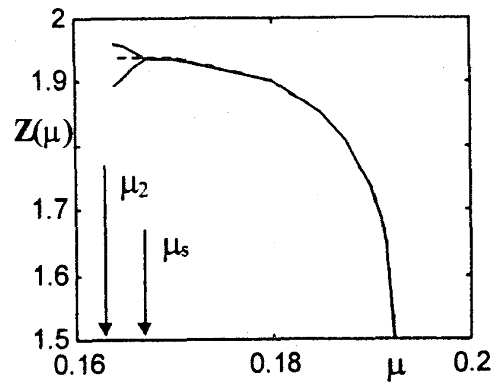

Figure 3: Simulation (solid line) and L_2 approximation (dashed line) of the periodic solution branch showing the prediction of period-doubling bifurcation (μ_2) with respect to the real value (μ_s).

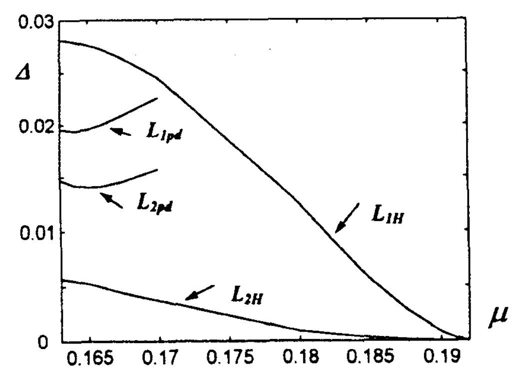

Figure 4: Distortion indexes for the predicted limit cycle and period-doubling orbit using L_1 and L_2 approximations.

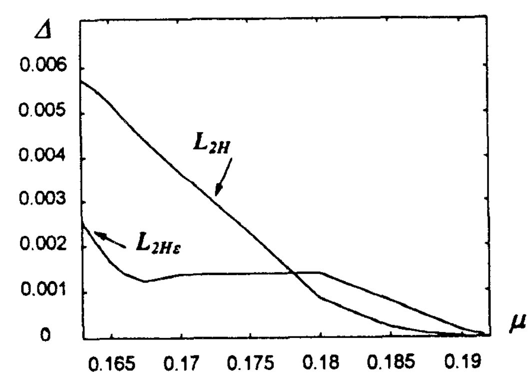

Figure 5: Distortion indexes for the predicted limit cycle and a period-doubling perturbation using L_2 approximation.